Appendix no. 4

SELF-PRESENTATION

1. Name: **Piotr Józef Bartłomiejczyk**

2. Obtained diplomas and degrees:

- M.Sc. in Mathematics, University of Gdańsk, Faculty of Mathematics, Physics and Chemistry, 1991;
- Ph.D. in Mathematics, Dissertation: *Topics in Connection Matrix Theory* (advisor prof. dr hab. K. Gęba), Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland, 2000.

3. Employment in scientific institutions:

- Assistant, Institute of Mathematics, University of Gdańsk, 1991–1999,
- Assistant Professor, Institute of Mathematics, University of Gdańsk, 2000–2011,
- Senior Lecturer, Institute of Mathematics, University of Gdańsk, 2011–2014,
- Senior Lecturer, Department of Differential Equations and Mathematics Application, Gdańsk University of Technology, from 2014.

4. Scientific achievements resulting from Article 16 Paragraph 2 of the Act on Academic Degrees and Titles and on Degrees and Titles in the Field of the Arts of 14 March 2003 is a series of publications under the title:

"Homotopy properties of the space of local maps"

List of publications included in the above mentioned achievement:

[H1] **P. Bartłomiejczyk**, K. Gęba, M. Izydorek, *Otopy classes of equivariant local maps*, J. Fixed Point Theory Appl. 7(1) (2010), 145–160.

[H2] **P. Bartłomiejczyk**, P. Nowak-Przygodzki, *Gradient otopies of gradient local maps*, Fund. Math. 214(1) (2011), 89–100.

[H3] **P. Bartłomiejczyk**, P. Nowak-Przygodzki, *Proper gradient otopies*, Topol. Appl. 159 (2012), 2570–2579.

[H4] **P. Bartłomiejczyk**, P. Nowak-Przygodzki, *The exponential law for partial, local and proper maps and its application to otopy theory*, Commun. Contemp. Math. 16(5) (2014), 1450005 (12 pages).

[H5] **P. Bartłomiejczyk**, P. Nowak-Przygodzki, *On the homotopy equivalence of the spaces of proper and local maps*, Cent. Eur. J. Math. 12(9) (2014), 1330–1336.

[H6] **P. Bartłomiejczyk**, *On the space of equivariant local maps*, Topol. Methods Nonlin. Anal. 45(1) (2015), 233–246.

[H7] **P. Bartłomiejczyk**, P. Nowak-Przygodzki, *The Hopf theorem for gradient local vector fields on manifolds*, New York J. Math. 21 (2015), 943–953.

Below is a discussion of the scientific aim of the above mentioned achievement.

1. Discussion of the results of the monothematic series of publications [H1]–[H7]

1.1. **Introduction.** The subject of works that make up the dissertation has its source in two currents of research. The first one concerns spaces of local maps and otopies and the second one is related to looking for new topological invariants in the class of gradient maps and homotopies.

The idea of studying spaces of partial, local and proper maps comes from [1,26, 27,33,42]. The paper [1] by A. M. Abd-Allaha and R. Brown from 1980 is the oldest and most elementary of them. The authors introduced there the space of partial maps $\mathrm{Par}(X, Y)$, where X, Y are topological spaces. This space consists of continuous maps $f\colon U \subset X \to Y$ defined on open subsets $U \subset X$ and its topology is a version of compact-open topology adapted to changing domains. Since, as it is easy to see, the above space is contractible if Y is contractible, not the whole space but its subsets consisting of local and proper maps have been used in nonlinear analysis. These subsets are also topological spaces but their topologies are essentially finer than the topology induced from the space of partial maps. Both spaces owe their usefulness to these topologies.

The space of proper maps appears in the paper [27] by J. C. Becker and D. H. Gottlieb from 1999. The topology in the set of local maps was introduced in our paper [21] and then in full generality in [H5]. It should be emphasized that in [H5] we introduce a generalized definition of a local maps that includes both local maps in the old sense and proper maps.

However, much earlier than we managed to define the topology in the set of local maps the notion of a local maps and a very useful generalization of the concept of homotopy called otopy have been introduced and used in the papers by J. C. Becker and D. H. Gottlieb ([26]) and D. H. Gottlieb and G. Samaranayake ([42]). The main advantage of using these notions is that otopy relates local maps with not necessarily the same domain, because the domain of a map may change along otopy. What is important is that the topological degree is otopy invariant and otopy classes appear naturally in many classification results.

The second important inspiration of the works making up the dissertation is the study of gradient maps and homotopies, in particular, the article [52] by A. Parusiński from 1990, which is closely related to discoveries made in the previous decade. Namely, in the middle of eighties E. N. Dancer gave a definition of a new degree-type invariant for S^1-equivariant gradient maps ([32]). Since this new degree provides more information than the usual degree, one can obtain new bifurcation results.

In the eighties of the last century Prof. K. Gęba posed the following problem: is there a better invariant for gradient homotopies of gradient maps than the usual topological degree? In 1990 A. Parusiński [52] gave the negative answer. Namely, he proved that if two gradient vector fields on the unit disc D^n and nonvanishing in S^{n-1} are homotopic (have the same degree), then they are gradient homotopic.

It occurs that the problem posed by Prof. K. Gęba appears naturally if we consider local maps and their otopy classes. For that reason analysis and comparison of gradient and usual otopy classes occupies an important place in our research.

It is worth pointing out that independently of Becker and Gottlieb, the similar notion was also developed by Dancer, Gęba and Rybicki in the article [33] from 2005. Understandably, they use different terminology. Local maps are called compact pairs (a pair consists of a map and its domain) and otopies are called homotopies of compact pairs. The authors use these notions as tools for proving results on the classification of equivariant gradient otopy classes.

All papers included in the dissertation concern the space of local maps and their various subspaces (with the induced topology) consisting of gradient or equivariant maps. We focus on the study of otopy classes of local maps i.e. path-components of the above spaces. We also show classifications of various sets of otopy classes (usual, gradient, equivariant) and natural relations between different sets of otopy classes.

For the sake of clarity, we will divide our discussion into three parts. In the first part we focus on the study of the set of gradient otopy classes. The main results of this part concern the set of gradient local maps in $\mathbb{R}^n$ ([H2]), the set of proper gradient maps in $\mathbb{R}^n$ ([H3]) and the set of gradient local vector fields on a manifold ([H7]). The main topic of the second part is an introduction of the topology in the set of local maps, which allows us to interpret otopies as paths in the space of local maps (similarly as for homotopies) and establish the relation between the theory of otopy and homotopy. In this part we also explain the relation between the space of proper maps and the space of local maps (in the narrower sense) if we restrict ourselves to the Euclidean case. In turn in the third part we deal with equivariant local maps and their otopies. We present here a version of the equivariant degree theory formulated in the language of otopies ([H1]) and results concerning the decomposition of the set of equivariant otopy classes with respect to set of orbit types ([H6]).

Let us mention that problems of that type seems to be quite natural. G. Segal proves in [58] that inclusions of some function spaces are homotopy (homology) equivalences. Similar results are contained in papers on configuration spaces by G. Segal and D. McDuff (see [50, 59]). On the other hand M. Gromov in his book ([43]) outlines a program of research on relations between a space of all maps and its subspaces given by some partial differential relations. Let us note that Schwarz condition (being such a differential relation) is equivalent to the statement that a map is gradient.

1.2. **Discussion of the publications [H2], [H3] i [H7].** The mentioned three articles are closely related. The main result of [H2] is the following version of the Parusiński theorem: the inclusion of the set of gradient local maps into the set of all local maps induces a bijection between the respective otopy classes of local maps. In other words, there is no better invariant in gradient otopy theory than the usual topological degree.

We expected that the analogous result should hold also for proper maps. The main advantage of using proper local maps instead of all local maps is that the space of proper local maps is a "very nice" metrizable space. In fact, it is homeomorphic to the space of based continuous maps of the n-sphere into itself. However, it turns out that the proof of of the Parusiński theorem for proper maps is more difficult compared to that concerning all local maps presented in [H2]. Although the main line of the proof is similar in both cases, the case of proper maps requires to develop some new ideas to overcome many (mainly technical) difficulties.

Finally, the complete proof of the theorem that the inclusion of the space of proper gradient local maps into the space of all proper local maps induces a bijection between the sets of connected components of these spaces i.e. the sets of the respective otopy classes of local maps appeared in [H3].

In [H2] and [H3] we have studied local and proper maps defined on open subsets of $\mathbb{R}^n$ and taking values in $\mathbb{R}^n$, while the main goal of [H7] has been to generalize the main result of [H2] to the case of an arbitrary Riemannian manifold without boundary. That generalization is not art for art's sake, because such a situation that is the case of gradient local vector fields on manifolds appears in a natural way in the analysis of equivariant gradient local maps. Namely, let V be an orthogonal representation of a compact Lie group G and Ω be an open invariant subset of V, on which G acts freely. Then there is a natural bijection between the set of otopy classes of equivariant gradient local maps in Ω and the set of otopy classes of gradient local vector fields on the manifold Ω/G.

To formulate precisely the above results let us introduce the following definitions. A continuous map $f\colon D_f \to \mathbb{R}^n$ is called *local* if D_f is an open subset of $\mathbb{R}^n$ and $f^{-1}(0)$ is compact. A local map f is called *gradient* if there is a C^1-function $\varphi\colon D_f \to \mathbb{R}$ such that $f = \nabla\varphi$ and *proper* if preimages of compact sets are compact.

Consider the set of all local maps $\mathcal{F}(n)$ and its following subsets:

$$\begin{aligned}\mathcal{F}_\nabla(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is gradient}\},\\ \mathcal{P}(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is proper}\},\\ \mathcal{P}_\nabla(n) &:= \mathcal{F}_\nabla(n) \cap \mathcal{P}(n).\end{aligned}$$

Let $I = [0,1]$. A continuous map $h\colon \Lambda \to \mathbb{R}^n$ is called an *otopy* if Λ is an open subset of $I \times \mathbb{R}^n$ and $h^{-1}(0)$ is compact. Given an otopy h we can define for each

$t \in I$ sets $\Lambda_t = \{x \in \mathbb{R}^n \mid (t,x) \in \Lambda\}$ and maps $h_t\colon \Lambda_t \to \mathbb{R}^n$ with $h_t(x) = h(t,x)$. Observe that h_t may be an empty map.

If h is an otopy, we say that h_0 i h_1 are *otopic*. Of course, otopy gives an equivalence relation on $\mathcal{F}(n)$. The set of otopy classes is denoted by $\mathcal{F}[n]$. Observe that if f is a local map and U is an open subset of D_f sych that $f^{-1}(0) \subset U$ then f and $f|_U$ are otopic. In particular, if $f^{-1}(0) = \emptyset$ then f is otopic to the empty map.

Apart from usual otopies we will consider otopies that satisfy some additional conditions, namely

- *gradient* i.e. $h(t,x) = \nabla_x \chi(t,x)$ for some not necessarily continuous C^1-function χ such that χ_t for each $t \in I$,
- *proper* i.e. h is proper,
- *proper gradient*.

The sets of the respective otopy classes in $\mathcal{F}_\nabla(n)$, $\mathcal{P}(n)$, $\mathcal{P}_\nabla(n)$ will be denoted by $\mathcal{F}_\nabla[n]$, $\mathcal{P}[n]$, $\mathcal{P}_\nabla[n]$.

Let us explain now why $\mathcal{P}(n)$, as opposite to $\mathcal{F}(n)$, has a natural structure of a metric space. Let $\Sigma^n = R^n \cup \{*\}$ be a one-point compactification of $\mathbb{R}^n$. It is a pointed space with the base point $*$. We write $\mathcal{M}_*\Sigma^n$ for the set of pointed continuous maps from Σ^n into Σ^n. With every map $f \in \mathcal{M}_*\Sigma^n$ one associates a proper map $f\restriction_{f^{-1}(R^n)}$. Conversely, if $f \in \mathcal{P}(n)$, then the function $f^+\colon \Sigma^n \to \Sigma^n$ given by

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in U, \\ * & \text{otherwise} \end{cases}$$

is continuous. It follows that the function $\mu\colon \mathcal{P}(n) \to \mathcal{M}_*\Sigma^n$ given by

$$\mu(f) = f^+$$

is a bijection. Since $\mathcal{M}_*\Sigma^n$ is equipped with the supremum metric, $\mathcal{P}(n)$ also has the metric structure induced by the pullback.

It is easy to see that the inclusion of the respective sets of the maps induce the following commutative diagram of the respective sets of otopy classes:

$$(*) \qquad \begin{array}{ccc} \mathcal{P}_\nabla[n] & \xrightarrow{\ \mathfrak{a}\ } & \mathcal{P}[n] \\ {\scriptstyle \mathfrak{b}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{c}} \\ \mathcal{F}_\nabla[n] & \xrightarrow{\ \mathfrak{d}\ } & \mathcal{F}[n] \end{array}$$

The briefest summary of the most important results from [H2] and [H3] can be formulated as follows.

Theorem 1 ([H2],[H3]). *All the functions in the diagram* $(*)$ *are bijections.*

In [H2] we showed that functions $\mathfrak{a}$ and $\mathfrak{b}$ are surjections and $\mathfrak{c}$ and $\mathfrak{d}$ are bijections. It is worth pointing out that our result includes a version of the Parusiński theorem: the function $\mathfrak{d}\colon \mathcal{F}_\nabla[n] \to \mathcal{F}[n]$ induced by the inclusion $\mathcal{F}_\nabla(n) \hookrightarrow \mathcal{F}(n)$ is a bijection. However, our proof makes no appeal to the original proof of Parusiński. It seems that our proof is simpler as an effect of replacing fixed domain and homotopies by local maps and otopies. The true difficulty in proving Theorem 1 lies in the following version of the Hopf theorem for gradient local maps ($\deg$ denotes the classical topological degree).

Theorem 2 ([H2]). *The function* $\deg\colon \mathcal{F}_\nabla[n] \to \mathbb{Z}$ *is bijective.*

Moreover, only injectivity causes here a problem.

In [H3] we present the essential strengthening of results from [H2]. Namely, we show that the functions $\mathfrak{a}$ and $\mathfrak{b}$ are also bijections. The general scheme of reasoning is similar in both cases. Also here the main difficulty lies in the proof of the Hopf type theorem, which says that the function $\deg\colon \mathcal{P}_\nabla[n] \to \mathbb{Z}$ is bijective. However, in the case of proper maps we have encountered a number of technical difficulties requiring the introduction of new notions and development of new ideas. Observe that the fact that $\mathfrak{a}$ is bijective may be treated as a version of the Parusiński theorem in the class of proper gradient maps.

In [H2] one another class of maps appears, namely proper gradient-like maps. The main reason for the introduction of this class is that in [H2] we were not able to prove that the function $\mathfrak{a}$ is bijective. Because of that we tried to define a class of maps that is similar to proper gradient, but in which we are able to prove the Parusiński type theorem. Of course, in light of the results of [H3] this class has lost its meaning.

We formulate now the main results of [H7]. The above mentioned definitions of (gradient) local maps and gradient otopies can be generalized to the case of a (Riemannian) manifold. Assume that M is a connected Riemannian manifold without boundary. Let $\mathcal{F}[M]$ ($\mathcal{F}^\nabla[M]$) denote the set of (gradient) otopy classes of (gradient) local vector fields on M and I denote the intersection number. It is easy to see that the intersection number (similarly as the topological degree) is constant on the otopy classes.

The main result of [H7] is the following Hopf type theorem.

Theorem 3 ([H7]). *The function* $\mathrm{I}\colon \mathcal{F}^\nabla[M] \to \mathbb{Z}$ *is bijective.*

Observe that the inclusion of the space of gradient local vector fields into the space of all local vector fields induces well-defined function $\Phi\colon \mathcal{F}^\nabla[M] \to \mathcal{F}[M]$. The following generalization of the Parusiński theorem is an immediate consequence of Theorem 3.

Corollary 4 ([H7]). *The function* Φ *is bijective.*

Apart from the mentioned results the paper [H7] contains their application to studying equivariant local maps, which was the main motivation of the above generalization.

Assume that V is a real finite dimensional orthogonal representation of a compact Lie group G, Ω is an open invariant subset of V, G acts freely on Ω and $M := \Omega/G$. It is well-known that M is a Riemannian manifold.

Let $\mathcal{F}_G[\Omega]$ ($\mathcal{F}_G^\nabla[\Omega]$) denote the set of equivariant (gradient) otopy classes of equivariant (gradient) local maps. Precise definitions of these notions from [H7] formulated using topology in the space of local maps introduced in [H4] are here omitted.

If U is an open invariant subset of Ω and $\varphi\colon U \to \mathbb{R}$ is invariant, then $\widetilde{\varphi}$ denotes the quotient function $\widetilde{\varphi}\colon U/G \to \mathbb{R}$. Let the function $\Psi\colon \mathcal{F}_G^\nabla[\Omega] \to \mathcal{F}^\nabla[M]$ be given by $\Psi([\nabla\varphi]) = [\nabla\widetilde{\varphi}]$. We present now to results from [H7] concerning the equivariant gradient case.

Theorem 5 ([H7]). *The function Ψ is bijective.*

The following result is an immediate consequence of Theorems 3 and 5.

Corollary 6 ([H7]). *There is a natural bijection*

$$\mathcal{F}_G^\nabla[\Omega] \approx \sum_\alpha \mathbb{Z},$$

where the direct sum is taken over the set of all connected components α of the manifold Ω/G.

The paper [H7] ends with the remark concerning the difference between the sets of gradient equivariant and equivariant otopy classes. Namely, in [13] we proved that there is a bijection $\mathcal{F}_G[\Omega] \approx \sum_\alpha \mathbb{Z}$, with the direct sum taken over all connected components of M, but only if $\dim G = 0$. If $\dim G > 0$, then the set $\mathcal{F}_G[\Omega]$ is trivial i.e. consists of one element. Consequently, the map $\mathcal{F}_G^\nabla[\Omega] \to \mathcal{F}_G[\Omega]$ induced by the inclusion is a bijection for $\dim G = 0$, but the sets $\mathcal{F}_G^\nabla[\Omega]$ and $\mathcal{F}_G[\Omega]$ are essentially different for $\dim G > 0$. Therefore the analogy with the Parusiński result ([52]) occurs only if $\dim G = 0$.

1.3. **Discussion of the publications [H4] and [H5].** The main goal of [H4] is the introduction of the topology on the set of local maps and the proof of the exponential law for local and proper maps. Moreover, we show that the inclusion of the space of proper maps into the space of local maps is a weak homotopy equivalence if we restrict ourselves to local maps with domains in $\mathbb{R}^{n+k}$ and ranges in $\mathbb{R}^n$.

In turn, the main result of [H5] says that the above spaces are not homotopy equivalent for $n > 1$. The case $n = 1$ still remains an open problem.

It is worth pointing out that the first attempt to define the topology in the space of local maps has been made in [21]. In this paper we also prove the exponential law and then we use it to describe (up to a homeomorphism) the space of local maps as a usual mapping space i.e. the space of maps with one fixed domain.

Quick after publishing [21] we have realized that the approach presented in [21], although logically correct, has two important limitations. First, it turns out that the class of local maps from [21] may be extended in such a way that it covers also both partial and proper maps. Moreover, also topology in the space of local maps (in a new broader sense) can be defined in a way that allows to obtain spaces of partial (see [1]) and proper (see [H2]) maps as particular (in some sense extreme) cases of the space of local maps in the generalized sense. That gives a broader view on otopy theory and allows one to avoid consideration of separate cases. Secondly, it turns out that a simple reversal (in relation to [21]) of the order of proving theorems, namely first the description of the space of local maps as the usual mapping space and then the proof of the exponential law, essentially simplifies proofs, because it allow us to use the standard exponential law i.e. for maps with one fixed domain. For that reason we have decided to present this new both generalized (definitions) and simplified (proofs) approach in [H4]. Furthermore, [H4] contains additional sections concerning otopies in Euclidean spaces.

Let us now turn to necessary definitions and precise formulation of the main results of [H4] and [H5]. The notation $A \Subset B$ means that A is a compact subset of B. For topological space X, we denote by $\tau(X)$ the topology on X. Recall that if A, B are topological spaces, then $\operatorname{Map}(A, B)$ denotes the set of all continuous maps of A into B equipped with the usual compact-open topology i.e. having as subbasis all the sets $\Gamma(C, U) = \{ f \in \operatorname{Map}(A, B) \mid f(C) \subset U \}$, where $C \Subset A$ and $U \in \tau(B)$. Moreover, for any pointed topological spaces A and B, let $\operatorname{Map}_*(A, B)$ be the subspace of $\operatorname{Map}(A, B)$ consisting of all base-point preserving maps.

For any topological space X and $* \notin X$, let $\widetilde{X}$ denote the set $X \cup \{*\}$. Below we will use different topologies on $\widetilde{X}$. Let $\mathcal{R}$ be a family of subsets of X. We will denote by $X^+_{\mathcal{R}}$ the set $\widetilde{X}$ with the topology generated by the subbasis $\mathcal{S} := \tau(X) \cup \{\widetilde{X}\} \cup \{\widetilde{X} \setminus R \mid R \in \mathcal{R}\}$. If $\mathcal{R} = \emptyset$, we will abbreviate $X^+_{\mathcal{R}}$ to X^+ and if $\mathcal{K} := \{K \mid K \Subset X\}$, we will abbreviate $X^+_{\mathcal{K}}$ to X^*. Observe that if X is Hausdorff, then X^* is the familiar one-point compactification.

For any topological spaces X and Y, let $\operatorname{Par}(X, Y)$ be the set of all continuous maps $f\colon D_f \to Y$ such that D_f is an open subset of X. Elements of $\operatorname{Par}(X, Y)$ are called *partial maps*. We introduce the compact-open topology in $\operatorname{Par}(X, Y)$ i.e. generated by the sets $H(C, U) = \{ f \in \operatorname{Par}(X, Y) \mid C \subset D_f,\ f(C) \subset U \}$ for $C \Subset X$ and $U \in \tau(Y)$ as its subbasis. Note that $\operatorname{Par}(X, Y)$ is not T_1, since the only neighborhood of the empty map is the whole space $\operatorname{Par}(X, Y)$.

Let X, Y be any topological spaces and $\mathcal{R}$ a family of subsets of Y. We define

$$\mathrm{Loc}(X,Y,\mathcal{R}) := \{\, f \in \mathrm{Par}(X,Y) \mid f^{-1}(R) \Subset D_f \text{ for all } R \in \mathcal{R} \,\}.$$

We introduce a topology in $\mathrm{Loc}(X,Y,\mathcal{R})$ generated by the subbasis consisting of all sets of the form

- $H(C,U) := \{\, f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid C \subset D_f,\ f(C) \subset U \,\}$ for $C \Subset X$ i $U \in \tau(Y)$,
- $M(V,R) := \{\, f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid f^{-1}(R) \subset V \,\}$ for $V \in \tau(X)$ i $R \in \mathcal{R}$.

Elements of $\mathrm{Loc}(X,Y,\mathcal{R})$ will be called *local maps*. The natural base point of $\mathrm{Loc}(X,Y,\mathcal{R})$ is the empty map.

We can now formulate the main results of [H4]. Let us start with the description of the space of local maps as the usual mapping space.

Proposition 1 ([H4]). *If X is locally compact Hausdorff, then the function*

$$\kappa\colon \mathrm{Loc}(X,Y,\mathcal{R}) \to \mathrm{Map}_*(X^*, Y^+_{\mathcal{R}})$$

given by $\kappa(f) := f^+$, where

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in D_f, \\ * & \text{otherwise}, \end{cases}$$

is a homeomorphism.

Note that $\mathrm{Loc}(X,Y,\mathcal{R}) = \mathrm{Par}(X,Y)$ if $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{\emptyset\}$. In turn if $\mathcal{R} \neq \emptyset$ and $\mathcal{R} \neq \{\emptyset\}$ then the inclusion $\mathrm{Loc}(X,Y,\mathcal{R}) \hookrightarrow \mathrm{Par}(X,Y)$ is continuous, but the topology on $\mathrm{Loc}(X,Y,\mathcal{R})$ is finer than the induced topology.

In what follows we will be especially interested in the case when $\mathcal{R} = \{\{y\}\}$ with $y \in Y$. In this case we will write $\mathrm{Loc}(X,Y,y)$ omitting double curly brackets.

Recall that a map between topological spaces is called *proper* if preimages of compact subsets are compact.

Let X and Y be topological spaces. Define $\mathrm{Prop}(X,Y) := \mathrm{Loc}(X,Y,\mathcal{K})$, where $\mathcal{K} := \{K \mid K \Subset Y\}$. It is easily seen that (as sets)

$$\mathrm{Prop}(X,Y) = \{\, f \in \mathrm{Par}(X,Y) \mid f \text{ is proper} \,\}.$$

The next result is an immediate consequences of Proposition 1.

Proposition 2 ([H4]). *Assume that X is locally compact Hausdorff. Then the function*

$$\kappa\colon \mathrm{Prop}(X,Y) \to \mathrm{Map}_*(X^*, Y^*),$$

given by the same formula as in Proposition 1, is a homeomorphism.

We can now formulate the exponential law for local maps.

Theorem 3 ([H4]). *If Z and X are locally compact Hausdorff, then the exponential function*

$$\theta\colon \mathrm{Loc}(Z \times X, Y, \mathcal{R}) \to \mathrm{Map}_*(Z^*, \mathrm{Loc}(X, Y, \mathcal{R}))$$

given by $\theta(h) = h^*$, where $h^*(t)(x) = h(t, x)$, is a homeomorphism.

The following consequence of Theorem 3 is especially useful in applications.

Corollary 4 ([H4]). *If Z is compact Hausdorff and X is locally compact Hausdorff, then the exponential function*

$$\theta\colon \mathrm{Loc}(Z \times X, Y, \mathcal{R}) \to \mathrm{Map}\,(Z, \mathrm{Loc}(X, Y, \mathcal{R}))$$

is a homeomorphism.

We will now discuss briefly the results of [H4] and [H5] concerning local maps in euclidean spaces. Let us introduce the following notation:

$$\mathcal{F}(n, k) := \mathrm{Loc}(\mathbb{R}^{n+k}, \mathbb{R}^n, 0),$$

$$\mathcal{P}(n, k) := \mathrm{Prop}(\mathbb{R}^{n+k}, \mathbb{R}^n).$$

We will abbreviate $\mathcal{F}(n, 0)$ (resp. $\mathcal{P}(n, 0)$) to $\mathcal{F}(n)$ (resp. $\mathcal{P}(n)$). Moreover, we will denote by $\mathcal{F}_\alpha(n, k)$ (resp. $\mathcal{P}_\alpha(n, k)$) that component of $\mathcal{F}(n, k)$ (resp. $\mathcal{P}(n, k)$) which contains α (we write 0 for the empty map).

Let $S^n_0 := (\mathbb{R}^n)^+_{\mathcal{R}}$ with $\mathcal{R} = \{\{0\}\}$. Observe that, by Proposition 1, there are natural homeomorphisms

$$\mathcal{P}(n, k) \approx \Omega^{n+k}(S^n) \quad \text{and} \quad \mathcal{F}(n, k) \approx \Omega^{n+k}(S^n_0).$$

In particular, $\mathcal{P}(n) \approx \Omega^n(S^n)$ and $\mathcal{F}(n) \approx \Omega^n(S^n_0)$.

Relation between local and proper maps in Euclidean spaces is explained in the following two theorems. The first one has been proved in [H4] and the second [H5].

Theorem 5 ([H4]). *The inclusion $\mathcal{P}(n, k) \hookrightarrow \mathcal{F}(n, k)$ is a weak homotopy equivalence.*

Theorem 6 ([H5]). *If $n > 1$ and $k \geq 0$ then the spaces $\mathcal{P}_0(n, k)$ and $\mathcal{F}_0(n, k)$ are not homotopy equivalent.*

Topology in the space of local maps and the exponential law allows us to formulate clearly the basics of otopy theory in the euclidean case, which is a part of [H4].

Let $I = [0, 1]$. Any element of $\mathrm{Loc}(I \times \mathbb{R}^{n+k}, \mathbb{R}^n, 0)$ is called an *otopy* and any element of $\mathrm{Prop}(I \times \mathbb{R}^{n+k}, \mathbb{R}^n)$ is called a *proper otopy*. By the exponential law for local maps, each (proper) otopy corresponds to a path in $\mathcal{F}(n, k)$ ($\mathcal{P}(n, k)$) and vice versa.

Given a (proper) otopy $h\colon \Omega \to \mathbb{R}^n$ we can define for each $t \in I$ sets $\Omega_t = \{x \in \mathbb{R}^{n+k} \mid (x,t) \in \Omega\}$ and maps $h_t\colon \Omega_t \to \mathbb{R}^n$ with $h_t(x) = h(x,t)$. If h is a (proper) otopy, we say that h_0 and h_1 are *(proper) otopic*. Of course, (proper) otopy gives an equivalence relation on $\mathcal{F}(n,k)$ ($\mathcal{P}(n,k)$). The set of (proper) otopy classes will be denoted by $\mathcal{F}[n,k]$ ($\mathcal{P}[n,k]$).

The following fact is crucial in the proof of Theorem 5.

Proposition 7 ([H4]). *The function $\mathcal{P}[n,k] \to \mathcal{F}[n,k]$ induced by the inclusion is a bijection.*

By the exponential law, we obtain the following canonical isomorphisms:

$$\mathcal{P}[n,k+m] \approx \pi_m\left(\mathcal{P}_0(n,k)\right) \approx \pi_{m+k}(S^n),$$
$$\mathcal{F}[n,k+m] \approx \pi_m\left(\mathcal{F}_0(n,k)\right) \approx \pi_{m+k}(S^n_0)$$

for $m > 0$.

1.4. **Discussion of the publications [H1] and [H6].** In [H1] and [H6] we study sets of equivariant (usual and gradient) otopy classes of local maps in the case of a real finite dimensional orthogonal representation of a compact Lie group G. The article [H1] is the oldest in a series of publications constituting the dissertation and was published in 2010. The main aim of [H1] is presentation of some extensions of the topological degree to equivariant local maps both in the gradient and non-gradient case and explain the relation between these two generalizations.

In turn, in [H6] we introduce the space of equivariant local maps and study their basic propreties. In particular, we present the full proof of the splitting theorem for the set of otopy classes of such maps in the case of a representation of a compact Lie group.

We will give now the precise formulation of the main results of [H1] and [H6]. Let G be a compact Lie group. Any subgroup H of G is understood to be closed and (H) stands for conjugacy class of H. We denote by NH the normalizer of H in G and by WH the associated quotient group NH/H called the *Weyl group*. The following notation will also be used:

$$\Phi(G) = \{\,(H) \mid H \text{ is a closed subgroup of } G\,\},$$
$$\Phi_k(G) = \{\,(H) \in \Phi(G) \mid \dim WH = k\,\},$$
$$\Phi_{k,\beta}(G) = \{\,(H) \in \Phi_k(G) \mid WH \text{ is biorientable}\,\},$$
$$\Phi_{k,\nu}(G) = \{\,(H) \in \Phi_k(G) \mid WH \text{ is not biorientable}\,\}.$$

The free abelian group $U(G)$ generated by $\Phi(G)$ admits a natural ring structure described by T. tom Dieck in [61] and is called the Euler ring. Its subgroup $A(G)$ generated by $\Phi_0(G)$ with multiplication defined in a similar manner is called the Burnside ring. The natural projection

$$\iota\colon U(G) \to A(G)$$

is a group homomorphism that in general does not preserve the ring structure.

Finally, we define

$$A_k(G) := \bigoplus_{(H)\in\Phi_{k,\beta}(G)} \mathbb{Z} \;\oplus \bigoplus_{(H)\in\Phi_{k,\nu}(G)} \mathbb{Z}_2 \qquad \text{for } k = 0, 1, \ldots, \dim G.$$

Let V be a real orthogonal representation of a compact Lie group G and $\mathbb{R}^k$ denotes a trivial representation of G. A *local map* on $V \oplus \mathbb{R}^k$ is a pair (f, U) consisting of an open invariant $U \subset V \oplus \mathbb{R}^k$ and an equivariant continuous map $f\colon U \to V$ such that $f^{-1}(0)$ is compact. Denote by $\mathcal{F}(V \oplus \mathbb{R}^k)$ the set of all local maps on $V \oplus \mathbb{R}^k$. An *otopy* on $V \oplus \mathbb{R}^k$ is a pair (h, Ω) consisting of an open invariant $\Omega \subset (V \oplus \mathbb{R}^k) \times I$ and an equivariant continuous map $h\colon \Omega \to V$ such that $h^{-1}(0)$ is compact. Denote by $\mathcal{O}(V \oplus \mathbb{R}^k)$ the set of all otopies on $V \oplus \mathbb{R}^k$. In a special case $k = 0$ we say that $(f, U) \in \mathcal{F}(V)$ is a *gradient local map* if there is an invariant C^1 function $\varphi\colon U \to \mathbb{R}$ such that $f = \nabla\varphi$. Similarly, we say that $(h, \Omega) \in \mathcal{O}(V)$ is a *gradient otopy* if there is an invariant C^1 function $\psi\colon \Omega \to \mathbb{R}$ such that $h(x, t) = \nabla\psi_t(x)$, where $\psi_t(x) = \psi(x, t)$. We denote by $\mathcal{F}^\nabla(V)$ the subset of $\mathcal{F}(V)$ consisting of all gradient local maps and by $\mathcal{O}^\nabla(V)$ the subset of $\mathcal{O}(V)$ consisting of all gradient otopies. Given $(h, \Omega) \in \mathcal{O}(V \oplus \mathbb{R}^k)$ and $t \in [0, 1]$ let $(h, \Omega)_t := (h_t, \Omega_t)$, where $\Omega_t := \{x \in V \oplus \mathbb{R}^k; (x, t) \in \Omega\}$ and $h_t(x) = h(x, t)$. We say that (h, Ω) is an otopy from (h_0, Ω_0) to (h_1, Ω_1).

Let us formulate two main results of [H1].

Theorem 1 ([H1]). *There exist two functions:*

(a) $\mathbf{deg}_G$ *assigning to each* $(f, U) \in \mathcal{F}(V)$ *an element* $\mathbf{deg}_G(f, U) \in A(G)$;

(u) $\mathbf{deg}_G^\nabla$ *assigning to each* $(f, U) \in \mathcal{F}^\nabla(V)$ *an element* $\mathbf{deg}_G^\nabla(f, U) \in U(G)$.

These functions have the following properties.

(1) $\iota(\mathbf{deg}_G^\nabla(f, U)) = \mathbf{deg}_G(f, U)$ *for* $(f, U) \in \mathcal{F}^\nabla(V)$.

(2) (a) $\mathbf{deg}_G(h_0, \Omega_0) = \mathbf{deg}_G(h_1, \Omega_1)$ *for* $(h, \Omega) \in \mathcal{O}(V)$,
(u) $\mathbf{deg}_G^\nabla(h_0, \Omega_0) = \mathbf{deg}_G^\nabla(h_1, \Omega_1)$ *for* $(h, \Omega) \in \mathcal{O}^\nabla(V)$.

(3) *Suppose* $U_1 \cap U_2 = \emptyset$. *Then*
(a) *if* $(f_1, U_1), (f_2, U_2) \in \mathcal{F}(V)$, *then*

$$\mathbf{deg}_G(f_1 \sqcup f_2, U_1 \cup U_2) = \mathbf{deg}_G(f_1, U_1) + \mathbf{deg}_G(f_2, U_2),$$

(u) *if* $(f_1, U_1), (f_2, U_2) \in \mathcal{F}^{\nabla}(V)$, *then*

$$\mathbf{deg}_G^{\nabla}(f_1 \sqcup f_2, U_1 \cup U_2) = \mathbf{deg}_G^{\nabla}(f_1, U_1) + \mathbf{deg}_G^{\nabla}(f_2, U_2),$$

where $(f_1 \sqcup f_2)(x) = f_1(x)$ *for* $x \in U_1$ *and* $(f_1 \sqcup f_2)(x) = f_2(x)$ *for* $x \in U_2$.

(4) *Let* $f\colon U \to V$ *be* C^1. *Suppose that* $f^{-1}(0) = Ga$ *for some* $a \in U$ *and* $Df(a)(v) = v$ *for all* $v \in (T_a(Ga))^{\perp}$. *Then*

(a)

$$\mathbf{deg}_G(f, U) = \begin{cases} (G_a), & \textit{if} \ (G_a) \in \Phi_0(G), \\ 0, & \textit{if} \ (G_a) \notin \Phi_0(G), \end{cases}$$

(u)

$$\mathbf{deg}_G^{\nabla}(f, U) = (G_a).$$

Theorem 2 ([H1]). *There exists a function* $\mathbf{deg}_G^k$ *assigning to each* $(f, U) \in \mathcal{F}(V \oplus \mathbb{R}^k)$ *an element* $\mathbf{deg}_G^k(f, U) \in A_k(G)$ *such that*

(1) $\mathbf{deg}_G^k(h_0, \Omega_0) = \mathbf{deg}_G^k(h_1, \Omega_1)$ *for* $(h, \Omega) \in \mathcal{O}(V \oplus \mathbb{R}^k)$.

(2) *If* $(f_1, U_1), (f_2, U_2) \in \mathcal{F}(V \oplus \mathbb{R}^k)$ *and* $U_1 \cap U_2 = \emptyset$ *then*

$$\mathbf{deg}_G^k(f_1 \sqcup f_2, U_1 \cup U_2) = \mathbf{deg}_G^k(f_1, U_1) + \mathbf{deg}_G^k(f_2, U_2),$$

where $(f_1 \sqcup f_2)(x) = f_1(x)$ *for* $x \in U_1$ *and* $(f_1 \sqcup f_2)(x) = f_2(x)$ *for* $x \in U_2$.

(3) *Let* $f\colon U \to V$ *be* C^1. *Suppose that* $f^{-1}(0) = Ga$ *for some* $a \in U$ *and* $Df(a)(v) = v$ *for all* $v \in (T_a(Ga))^{\perp}$. *Then*

$$\mathbf{deg}_G^k(f, U) = \begin{cases} (G_a), & \textit{if} \ (G_a) \in \Phi_k(G), \\ 0, & \textit{if} \ (G_a) \notin \Phi_k(G). \end{cases}$$

We will present now main results of [H6]. Let us start with some notation. Assume V is a real finite dimensional orthogonal representation of a compact Lie group G and H is a closed subgroup of G. Recall that $G_x = \{g \in G \mid gx = x\}$, (H) stands for a conjugacy class of H and $WH = NH/H$, where NH is a normalizer of H in G. Let Ω be an open invariant subset of V. We define the following subsets of Ω:

$$\Omega^H = \{x \in \Omega \mid H \subset G_x\},$$
$$\Omega_H = \{x \in \Omega \mid H = G_x\},$$
$$\Omega_{(H)} = \{x \in X \mid (H) = (G_x)\}.$$

Let

$$\Phi(G) = \{(H) \mid H \text{ is a closed subgroup of } G\},$$
$$\mathrm{Iso}(\Omega) = \{(H) \in \Phi(G) \mid \Omega_{(H)} \neq \emptyset\}.$$

The set $\mathrm{Iso}(\Omega)$ is partially ordered. Namely, $(H) \leq (K)$ if H is conjugate to a subgroup of K. Below we will make use of the following well-known facts:

- $\mathrm{Iso}(\Omega)$ is finite,
- WH is a compact Lie group,
- V^H is a linear subspace of V and orthogonal representation of WH,
- the action of WH on Ω_H is free,
- Ω_H is open and dense in Ω^H,
- $\Omega_{(H)}$ is a G-invariant submanifold of Ω,
- $\Omega_{(H)} = G\Omega_H$ and Ω_H is closed in $\Omega_{(H)}$,
- if (H) is maximal in $\mathrm{Iso}(\Omega)$ then $\Omega_{(H)}$ is closed in Ω.

Recall that if X, Y are topological spaces and $\mathcal{R}$ is a family of subsets of Y then $\mathrm{Loc}(X, Y, \mathcal{R})$ ($\mathrm{Prop}(X, Y)$) denotes the space of (proper) local maps introduced in [H4]. Due to [H4] we were able to formulate clearly the basics of otopy theory in the equivariant case. Namely, assume X, Y are G-spaces. Let $\mathrm{Loc}_G(X, Y, \mathcal{R})$ (resp. $\mathrm{Prop}_G(X, Y)$) be the subspace of $\mathrm{Loc}(X, Y, \mathcal{R})$ (resp. $\mathrm{Prop}(X, Y)$) consisting of equivariant maps with invariant domains and equipped with the induced topology.

Let Ω be an open invariant subset of $\mathbb{R}^k \oplus V$. Let us introduce the following notation:

$$\mathcal{F}_G(\Omega) := \mathrm{Loc}_G(\Omega, V, 0),$$
$$\mathcal{P}_G(\Omega) := \mathrm{Prop}_G(\Omega, V).$$

Let $I = [0, 1]$. We assume that the action of G on I is trivial. Any element of $\mathrm{Loc}_G(I \times \Omega, V, 0)$ is called an *otopy* and any element of $\mathrm{Prop}_G(I \times \Omega, V)$ is called a *proper otopy*.

Given a (proper) otopy $h\colon \Lambda \subset I \times \Omega \to V$ we can define for each $t \in I$ sets $\Lambda_t = \{x \in \Omega \mid (t, x) \in \Lambda\}$ and maps $h_t\colon \Lambda_t \to V$ with $h_t(x) = h(t, x)$. Note that from the above h_t may be the empty map. If h is a (proper) otopy, we say that h_0 and h_1 are *(proper) otopic*. Of course, (proper) otopy gives an equivalence relation on $\mathcal{F}_G(\Omega)$ ($\mathcal{P}_G(\Omega)$). The set of (proper) otopy classes will be denoted by $\mathcal{F}_G[\Omega]$ ($\mathcal{P}_G[\Omega]$).

At the beginning of [H6] we formulate the following results concerning the sets of equivariant otopy classes.

Theorem 3 ([H6]). *The function $\mathcal{P}_G[\Omega] \to \mathcal{F}_G[\Omega]$ induced by the inclusion is a bijection.*

Theorem 4 ([H6]). *The inclusion $\mathcal{P}_G(\Omega) \hookrightarrow \mathcal{F}_G(\Omega)$ is a weak homotopy equivalence.*

Theorem 5 ([H6]). *If $\dim G > 0$, Ω is an open invariant subset of V and G acts freely on Ω then the set $\mathcal{F}_G[\Omega]$ has a single element.*

The next main goal of [H6] was to show that, under the assumption that (H) is maximal in $\mathrm{Iso}(\Omega)$, there is a natural bijection between the sets $\mathcal{F}_G[\Omega]$ and $\mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G\left[\Omega \setminus \Omega_{(H)}\right]$. The naive approach suggests to define this bijection simply by taking the otopy classes of the respective restrictions i.e. by the formula

$$[f] \mapsto \left(\left[f\restriction_{D_f \cap \Omega_H} \right], \left[f\restriction_{D_f \setminus \Omega_{(H)}} \right] \right).$$

Unfortunately, $f\restriction_{D_f \setminus \Omega_{(H)}}$ does need not to be a local G-map. For this reason, we first have to perturbate the map f within its otopy class so that the restriction of the perturbation to the set $D_f \setminus \Omega_{(H)}$ would be a local G-map. Roughly speaking, our perturbation does not change f on $\Omega_{(H)}$ and separates zeros of maximal orbit type, which lie on $\Omega_{(H)}$, from all other zeros of f. The precise definition of this bijection from [H6] will be omitted here, since it requires much more additional notation and definitions. However, let us formulate the result.

Theorem 6 ([H6]). *If (H) is maximal in $\operatorname{Iso}(\Omega)$ then there is a natural bijection $\Theta\colon \mathcal{F}_G[\Omega] \to \mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G\left[\Omega \setminus \Omega_{(H)}\right]$.*

The paper ends [H6] with a series of splitting results, which are consequences of Theorem 6. Assume V is a real finite dimensional orthogonal representation of a compact Lie group G and Ω is an open invariant subset of $\mathbb{R}^k \oplus V$. Let $\operatorname{Iso}_k(\Omega) := \{(H) \in \operatorname{Iso}(\Omega) \mid \dim WH \leq k\}$. It is well-known that the set $\operatorname{Iso}(\Omega)$ is finite and so is $\operatorname{Iso}_k(\Omega)$. Let us denote by S^{k+V} and S^V *representation spheres* i.e. one-point compactifications of representations $\mathbb{R}^k \oplus V$ and V, respectively. The set of G-homotopy classes of such maps will be denoted by $[S^{k+V}; S^V]^*_G$. Recall that if X, Y are G-spaces and A (resp. B) is a G-subspace of X (resp. Y) then the set of relative G-homotopy classes of G-maps from (X, A) to (Y, B) is denoted by $[X, A; Y, B]_G$. The sets of (proper) otopy classes of equivariant (proper) local maps can be identified with the sets of path-components of the spaces of equivariant (proper) local maps.

Theorem 7 ([H6]). *There are natural bijections*

(1) $$\mathcal{F}_G[\Omega] \approx \prod_{(H)} \mathcal{F}_{WH}[\Omega_H],$$

(2) $$\mathcal{P}_G[\Omega] \approx \prod_{(H)} \mathcal{P}_{WH}[\Omega_H],$$

(3) $$\left[S^{k+V}; S^V\right]^*_G \approx \prod_{(H)} \left[S^{k+V^H}, S^{k+V^H} \setminus (\mathbb{R}^k \times V_H); S^{V^H}, *\right]_{WH},$$

where the products are taken over the set $\operatorname{Iso}_k(\Omega)$.

It should be emphasized that the main difficulty in proving Theorem 7 lies in Theorem 6.

Recall that the extreme case of the trivial action is covered by [H2] and [H4]. Namely, if G acts trivially on V and Ω is an open subset of V, then

$$\mathcal{F}_G[\Omega] = \mathcal{F}_{\{e\}}[\Omega] \approx \sum_{\alpha} \mathbb{Z},$$

where the direct sum is taken over all connected components α of the set Ω. Similarly, if G acts trivially on $\mathbb{R}^{n+k}$, then

$$\mathcal{F}_G\left[\mathbb{R}^{n+k}\right] = \mathcal{F}_{\{e\}}\left[\mathbb{R}^{n+k}\right] \approx \pi_{n+k}(S^n).$$

A more thorough description of $\mathcal{F}_G[\Omega]$ based on the formula (1) and the detailed analysis of the factors $\mathcal{F}_{WH}\left[\Omega_H\right]$ is given in [13], which continues and develops the approach presented in [H6].

2. Discussion of other scientific achievements

2.1. **Papers from Conley Index Theory.** Two papers preceding the Ph.D. dissertation ([14, 15]), the Ph.D. dissertation ([6]) and five papers published after it ([7–10, 12]) are devoted to the Conley index theory.

One of the main ideas behind the Conley index theory is to apply the tools from algebraic topology in studying dynamical systems, especially the structure of invariant sets (see [29, 30, 49, 57]). This approach, motivated by Morse theory, focuses on decomposing isolated invariant sets into invariant subsets (Morse sets) and connecting orbits between them. This structure is called a Morse decomposition of an isolated invariant set. A filtration of index pairs associated with a Morse decomposition can be used to find connections between different Morse sets. The principal tools for this purpose are connection matrices (see [9, 10, 38, 54]), connection graphs (see [8, 37]) and spectral sequences (see [7, 10, 31]).

The papers [14, 15] concern the connection matrix theory for discrete dynamical systems. The main goal of [14] is the proof of existence of index filtrations. Existence of such filtrations in the case of continuous dynamical systems (flows) has been proved in [30] and [57]. In [14] we present the proof in the discrete case when a dynamical system given by a homeomorphism of a locally compact metric space. In turn in [15] we carry over to the discrete case the construction of the connection matrix modifying the construction for flows from [38]. The main difference is that in the discrete case the homology Conley index is not simply the homology of the index pair but the Leray reduction of its homology. The papers [14, 15] have mainly the technical character and are preparation for the Ph.D. dissertation ([6]).

The paper [8] contains a generalized version of the main result of the Ph.D. dissertation i.e. theorem on existence of connection graphs. In the dissertation we prove it in the case of flows and in [8] we provide a parallel proof for both flows and homeomorphisms. Although the main line of the proof remains the same, some parts of the proof (mainly in the algebraic aspects) are presented in a new more clarified form. Moreover, [8] is extended by a series of examples illustrating obtained results.

In [7] we define spectral sequences associated with Morse decompositions of a compact metric space. We prove the existence and uniqueness of such spectral sequences for continuous dynamical systems.

The paper [9], similarly to [14, 15], concerns the theory of connection matrices. Recall that connection matrices can be seen as algebraic representations of the dynamical system and they express the relationship between certain (co)homology groups. Since the classical definition of the connection matrix is quite complicated, in [9] we introduce so called simple connection matrices, which are the simplest possible version of these algebraic tools and prove existence of such matrices for Morse decompositions of a compact metric space. This way we maintain the basic idea and avoid many technical details, which blur the overall picture.

The articles [10, 12] are the most mature works in a series devoted to the Conley index theory. In [10] we study the relation between spectral sequences and connection matrices. Recall that both spectral sequences and connection matrices are a generalization of exact sequences. The idea of the connection matrix was due to Charles Conley and the connection matrix theory was developed by his students. In [10] we introduce detailed connection matrices for filtered differential vector spaces. A filtered differential vector space is a finite increasing filtration of a given vector space together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration. Roughly speaking, a detailed connection matrix is a bigraded subspace of the filtered differential vector space which provides information on some homology groups associated with the filtered differential vector space. It is well known that similar information is contained in spectral sequences. Therefore, the main goal of [10] is to establish the clear and purely algebraic relation between detailed connection matrices and spectral sequences. More precisely, we prove that for a given filtered differential vector space there exist a detailed connection matrix that fully reconstructs its spectral sequence. The paper contains also some examples illustrating possible applications of the theory to dynamical systems.

The main goal of [12] is to explain and clarify the basic relations between connection matrices (generalized here to spectral splittings), connection graphs (called here spectral graphs) and spectral sequences. Although the comparison is done mainly on algebraic level, it sheds some new light on important aspects of the Conley index theory. This paper is also intended as a brief survey summarizing results from [7–10].

2.2. **Papers thematically similar to the series [H1]–[H7].** Several papers not included in the dissertation cover topics similar to that presented in [H1]–[H7]. It concerns the articles [19, 20] devoted to gradient vector fields on the two dimensional disc and the article [21], in which we introduce a first version of the definition of the topology in the set of local maps.

The papers [19, 20] address issues related to the Parusiński theorem ([52]). Recall that the Parusiński theorem can be formulated in the following way: the inclusion of the space of gradient vector fields in the space of all vector fields on D^n non-vanishing in S^{n-1} induces the bijection between the sets of path-components of these function spaces.

In [19] we strengthen the mentioned result for $n = 2$ via showing that the above inclusion is a homotopy equivalence (both spaces are homotopy equivalent to S^1). Precisely, this was partially proved in [52] using the argument of deformation retraction, but this method fails (at least in that form) in the case of the identity component. For that reason we have investigated in [19] this more difficult case.

In turn [20] contains a new proof of the Parusiński theorem in the case of the plane ($n = 2$). In our approach we wanted to emphasize strongly the geometric aspects of the proof. This approach was continued in [19]. Moreover, in [20] we filled a small gap in the original proof from [52].

The paper [21] has been discussed together with [H4]. Recall that the main aim of [21] was to introduce such a topology on the set of local maps in which otopies correspond to paths in this mapping space. Namely, we proved a version of the exponential law which establishes the homeomorphism between the space of otopies and the space of paths of local maps. The important point to note here is that the above-mentioned topology is essentially finer than the topology induced from the space of all partial maps.

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Piotr Bartłomiejczyk

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