Author's description of his scientific work

Marcin Szyszkowski

1 Education and profesional work

- 1992 - master in mathematics at Gdańsk University
- 2000 - Phd in mathematics at West Virginia University (USA), nostrificated 2001 at Łódz University

Emplyment

- 2000-2001 visiting professor position at Columbus State University, USA
- since 2001 work at Gdańsk University

2 Habilitation achievement

As such achievement I present a series of publications under the title

Axial functions

which consists of the following five papers:

[S1] M. Szyszkowski, *On axial maps of the direct product of finite sets*, Colloq. Math. 75 (1998), 31-38.

[S2] M. Szyszkowski, *A note on axial functions on the plane*, Tatra Mt. Math. Publ. 40 (2008), 59-62.

[S3] M. Szyszkowski, *Axial continuous functions*, Topology Appl. 157 (2010), 559-562.

[S4] M. Szyszkowski, *Axial Borel functions* Topology Appl. 160 (2013), 2049-2052 .

[S5] M. Szyszkowski, a chcapter *Axial functions* in a book *Traditional and present-day topics in real analysis*, Łódź 2013.

This is a cycle of papers concerning presenting a function of two variables as a composition of axial functions. A description of these papers is in the next sections

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2.1 Definition

Def. 1 *A function $f : X \times Y \to X \times Y$ is axial if it is one of two types*

$$f(x,y) = (x, g(x,y)) \text{ for some } g : X \times Y \to Y \quad (f \text{ is vertical})$$

or

$$f(x,y) = (g(x,y), y) \text{ for some } g : X \times Y \to X \quad (f \text{ is horizontal}).$$

The main question is which mappings f of the set $X \times Y$ are compositions of finnitely many axial mappings i.e. if $f = h_1 \circ ... \circ h_n$, where h_i are axial. Such quesions have been already stated by Banach and Ulam in Scottish Book [S]. Later Ulam in [U] posted another questions, on some of them I was able to answer. Part of article [S5] is a survey of axial functions.

2.2 Finite sets

To a question of Ulam [U, VIII 2] whether every function $f : X \times Y \to X \times Y$, where X i Y are finite, is a superposition of finitely many axial functions positive answer was given by Ehrenfeucht and Grzegorek in [EG]. The result from [EG] says that to obtain any function $f : X \times Y \to X \times Y$ six axial functions are enough, moreover we can demand the first axial function to be of wanted type e.g. vertical.

In this paper they state a question if one can decrease number six. In my article [S1] I decrease this number to five, and we still may demand the first axial function to be, say, vertical.

In the paper [S1] I give an example of a mapping (for $\|X\| = 5$ and $\|Y\| = 4$) which is not a composition of three axial functions (later Grzegorek and Płotka, see [P], found a similar example for $\|X\| = \|Y\| = 3$).

Płotka w [P] gave an example of a function $f : X \times Y \to X \times Y$ (where $|X| = 3$ and $|Y| = 93(!)$), which is not a composition of four axial functions provided that the first one is vertical, but is a composition of four axial functions if the first one is horizontal.

Problem if every function (for finite X and Y) is a composition of four axial mappings remains open.

The other result from [S1] (serving as a lemma to prove the main result about five functions) is : for any $f : X \times Y \to X \times Y$ there are axial functions h_1, h_2 i h_3 such that $|f^{-1}(x,y)| = |(h_3 \circ h_2 \circ h_1)^{-1}(x,y)|$ for all $(x,y) \in X \times Y$. This time number three can not be reduced.

Let us note ([EG]) that every permutation of $X \times Y$ (for finite X, Y) is a composition of only three axial permutations (and we may demand the first one to be, say, vertical).

2.3 Infinite sets

To a question of Banach [S, problem 47] whether every permutation of $\mathbb{N} \times \mathbb{N}$ can be obtained as a composition of some axial permutations a positive

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answer was obtained by Nosarzewska in [N]. Później in works [EG] and [G] this result has been generalised and strenghtened to the following theorems:

Thm. 2 [G] *Each permutation of $X \times Y$, where X or Y is infinite, is a composition of four axial permutations (however, we can not demand that the first one vertical).*

Thm. 3 [EG] *every function from $X \times Y$ into $X \times Y$, where X or Y is infinite, is a composition of three axial functions (not permutations).*

Both numbers (four and three) are minimal.

Let us note a contrast with finite sets where permutations are compositions of three axial permutations while for 'nonpermutations' it is still not known how many axial functions we need to compose.

Proofs of results from [S4] can be easily adapt (simlified) to obtain a result that permutations are compositions of eleven axial permutations. This result is of course weaker than the above one, nevertheless it is an alternative way of solving Banach's problem.

It is worth to mention a result of Komjath from [K] about very special axial permutations - slides (see next section) acting on $G \times G$, where G is an abelian group (infinite). The result says that every subset $A \subset G \times G$ can be transformed onto other set $B \subset G \times G$ by ten slides, provided that (quite obvious) conditions are satisfied : $|A| = |B|$ and $G \times G \setminus A| = G \times G \setminus B|$. A work of Abert [A] also using slides contains a result that a group of permutations of infinite group is a product od finitely many (289(!)) of some of its abelian subgropus.

2.4 The plane

A question of Ulam stated in [S, problem 20] if homeomorphism of a square $[0,1]^2$ can be approximated by compositions of axial homeomorphisms has been answered by Eggleston with the following theorems:

Thm. 4 [E] *Every homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a pointwise limit of $f_n : \mathbb{R}^2 \to \mathbb{R}^2$, where each$f_n$ is a composition of axial homemorphizms.*

Thm. 5 [E] *Let $f : [0,1]^2 \to [0,1]^2$ be a homeomorphism which is identity on the boundary of $[0,1]^2$ and let $\varepsilon > 0$. There exist axial homeomorphisms $h_i : [0,1]^2 \to [0,1]^2$, $i = 1,..,n$, such that $\forall_{p \in [0,1]^2}\ |h_n \circ \ldots \circ h_1(p) - f(p)| < \varepsilon$.*

So we can approximate arbitrary close (in supremum metric) homeomorphism of a square by composition of finitely many axial homeomorphisms. We can not, however, bound the number of axial homeomorphisms.
Although Eggleston does not give an example of homeomorphism of a square which can not be obtained 'exactly' as a composition of axial homeomorphisms, we can construct such an example see [S3].

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Knowing results of Eggleston, Ulam [U, IV 2] asked if we can generalise theorems 5 oraz 4 to continuous functions (not only homeomorphisms). This time the aswer is negative, namely in the paper [S3] we have

Thm. 6 *There is a continuous function $f : [0,1]^2 \to [0,1]^2$ being identity on the boundary of $[0,1]^2$ which can not be approximated in supremum metric by compositions of axial continuous functions better then $\frac{1}{10}$.*

Another question of Ulam [U, IV 2] was a question if we can obtain all Borel permutations)thus isomorphisms) as compositions of axial Borel permutations. This question also appeared in [S2]. The positive answer is in paper [S4]

Thm. 7 *Every Borel isomorphism of the plane is a composition of 11 axial Borel permutations. Additionally we may want the first axial permutation to be vertical.*

The number eleven probably is not minimal.

When we drop the requirement for axial functions to be permutations the situation simlifies a lot

Thm. 8 [S2] *Any Borel function from the plane to the plane is a composition of only three Borel axial function, additionally we may demand the first function to be, say, vertical.*

Number three is of course minimal. Clearly, the above axial functions are "far from being permutations" (they are not onto).

Let us note also a result from [AK], that every permutation of the plane is a composition of finitely many (209 (!)) of very special axial permutations - slides ($f : \mathbb{R}^2 \to \mathbb{R}^2$ is a slide if there exists $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = (x, y + g(x))$ or $f(x,y) = (x + g(y), y)$). In papers [S3] and [S4] we refer to the results in [AK].

2.5 Higher dimensions

Def. 9 [EG] *A function $f : X_1 \times \ldots \times X_n \to X_1 \times \ldots \times X_n$ is axial if there is $i \in \{1, \ldots, n\}$ and $g : X_1 \times \ldots \times X_n \to X_i$ satisfying*

$$f(x_1, \ldots, x_n) = (x_1, .., x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, .., x_n).$$

Thm. 10 [EG] *If at least one of the sets $X_1, \ldots, X_n$ is infinite then every $f : X_1 \times \ldots \times X_n \to X_1 \times \ldots \times X_n$ is a composition of $n+1$ axial functions $f = f_{n+1} \circ \ldots \circ f_1$.*

(the choice of f_1 is determined by which X_i has the greatest cardinality.)

Thm. 11 [EG] *For arbitrary sets $X_1, \ldots, X_n$ (finite or not) and any permutation $f : X_1 \times \ldots \times X_n \to X_1 \times \ldots \times X_n$ there is $k \in \mathbb{N}$ and axial permutations f_i so that $f = f_k \circ \ldots \circ f_1$.*

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In the case of $\mathbb{R}^n$ we may (like in previous section) include some topology. However, the only results we have concern Borel functions.
Theorem 8 can be easily generalised to $\mathbb{R}^n$, that is every Borel function is a composition of $n+1$ axial Borel functions. More interesting is, however, theorem from [S5] generalising Theorem 7

Thm. 12 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Borel isomorphism, ,then there is $k \in \mathbb{N}$ and axial Borel isomorphisms h_i with $f = h_k \circ \ldots \circ h_1$.*

E.g. for $n = 3$, $k \leq 22$, minimal numbers k (depending on n) are not known.

A counterpart of 5 for $[0,1]^3$ is propably true. For example using theorem of Bing and Moise ([B],[M]) that homeomorphisms of $[0,1]^3$ can be approximated with piecewise linear homeomorphisms and obtaining linear homeomorphism as compoisition of axial (linear) homeomorphisms looks promising.

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3 Other papers

Most (not all, thought) of other papers are devoted to symmetrically continuous functions and related topics. They are connected to my PhD thesis

Def. 13 *A funkcja $f : \mathbb{R} \to \mathbb{R}$ is symmetrically continuous at x if* $\lim_{h\to 0} f(x+h) - f(x-h) = 0$.

As we can see, it is a weaker condition than normal continuity, still symmetrically continuous functions poses many properties of continuous functions.

In the paper [CS] we give an example of (everywhere) symmetrically continuous functions which is not a 'countably continuous' function, that is the real line can not be splitted into countably many subsets so that on each of them our function is continuous. This answers a question of Larson [CL]

Articles [SD1] and [SD2] concern symmetrically continuous functions defined on subsets of the real line. It is proven there that if a domain is a measurable set or has the Baire property then symmetrically continuous function has similar properties as a function defined on the whole real line (or an interval). I also give examples of nonmeasurable domains where these properties are not preserved.

Papers [SS1], [SS2] and [SS3] treat about a weakening of symmetric continuity condition to a weaker condition

Def. 14 *A function $f : \mathbb{R} \to \mathbb{R}$ is weakly symmetrically continuous at a point x if there exists a sequence $h_n \to 0$ with* $\lim f(x + h_n) - f(x - h_n) = 0$.

I prove that arbitrary subset of the real line can be obtained as a set of points of weak symmetric continuity for some function (defined on the entire real line). Paper [SS2] is a strengthening of the paper [SS1]. In turn, paper [SS3] is a strengthening of the paper [SS2] and answers a question posted in [CNM] with a result that every subset of $\mathbb{R}$ is a set of points satisfying enhancement of Definition 14: there is a sequence $h_n \to 0$ with $\lim f(x+h_n) = f(x-h_n) = f(x)$.

An interesting result is in topological paper [MS]. It says that if we split a unit disk (on the plane) into two sets - one open and the other closed then one of them contains a connected component of diameter at least $\sqrt{3}$. As a corollary we get that every continuous real valued function from the disk has a level set also containing a connected component of diameter at least $\sqrt{3}$. Interestingly, for higher dimensions we have two instead of $\sqrt{3}$ - see [S] (and for the dimension one - zero).

Finally, [NMS] is a set thoretical work where we prove equivalence of CH with some property of measurable (in the Lebesgue sense) sets - which is a measure counterpart of cathegory result in [NW] and an answer to a question contained in [NW].

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