

Abstract

This dissertation concerns selected properties of ideals, σ -ideals, and related families of sets. The notions of porosity and σ -porosity play an important role and serve as a unifying theme for many of the topics under consideration. It also investigates direct sums of ideals of subsets of the natural numbers, representability by indexed systems of families of finite sets, and the existence of universal sets for ideals.

Porosity is a metric property of sets that is intuitively related to the presence of "holes" in a set. It is stronger than nowhere density. The term σ -porous refers to sets that are countable unions of porous sets. Over the years, numerous variants of porosity have been introduced, differing substantially in their definitions. A comprehensive overview of the subject can be found in the survey papers by Zajíček ([48], [47]).

The dissertation consists of six chapters. The first chapter collects the basic definitions, notation, and results from the literature used throughout the remainder of the dissertation. Definitions and notation specific to particular topics are introduced at the beginning of the corresponding chapters. This organization helps maintain the clarity of the exposition and limits the number of concepts introduced in the preliminary chapter.

Chapter 2 is based on the results of [31]. Its main objective is to construct a proper ideal on the Cantor space containing the Hindman ideal $\mathcal{H}$, the ideal of non-piecewise syndetic sets nPS , and the ideal of asymptotic density zero $\mathcal{I}_d$, which constitute the bottom part of the Kowitz Diagram [33, p. 20]. In this way, we complete the diagram and call the resulting ideal the Roman ideal, a name inspired by the saying *All roads lead to Rome*. An important part of the chapter is the analysis of the family of complements and the complement of the family of thick sets, that is, sets containing arbitrarily long intervals of natural numbers. We also investigate the properties of the operation Perm , obtaining new characterizations of the ideal nPS and solving problem posed in [31, Problem 1].

Chapter 3 uses the results established in the previous chapter and is based on [30]. It is devoted to σ -porosity of ideals on the Cantor space. We begin by reviewing the classical notions of porosity in metric spaces and then introduce their counterparts for families of subsets of the Cantor space. Firstly, we prove that no summable ideal is σ -porous. We then characterize σ -porous ideals in terms of the complement of the family of thick sets. In particular, we obtain that all ideals contained in this complement, including the Hindman ideal $\mathcal{H}$ and the ideal of non-piecewise syndetic sets nPS , are σ -porous.

Chapter 4 builds on the results of [38], extending them with new constructions and

properties – we introduce the amalgamation operation for indexed systems of families, discuss an example related to the Ellentuck topology, and solve Problem 3.6 posed in the aforementioned paper. We continue the study of the family of non-piecewise syndetic sets nPS by generalizing this notion to arbitrary spaces. Our approach is inspired by the classical Marczewski-Burstin scheme and the notion of MB-representability introduced in [34]. Using a known characterization of sets belonging to the ideal nPS , we construct a scheme in which the representation is given by an indexed system of families and the Marczewski-Burstin scheme is a special case. This construction leads to the notion of nPS -representability of ideals. In particular, we prove that every F_σ ideal is nPS -representable by an indexed system of families of finite sets.

Chapter 5 is based on the results of [39] and [32]. It concerns universal sets, which constitute a fundamental tool in descriptive set theory, used to study pointclasses and construct analytic non-Borel sets. In the first section, we compare various definitions of universal sets, both those known from the literature and those introduced in this dissertation. We then investigate the existence of closed universal sets, identifying both conditions for their existence and limitations of such constructions. Using the Vietoris topology, we also present an alternative scheme for constructing universal sets to that proposed in [15], which we apply to the family of porous sets and the σ -ideal of σ -porous sets. In the final part of the chapter, we move from universal sets for individual families to universal sets for classes of ideals. This leads to the introduction of the notion of a semiuniversal set and the first constructions for the family of all summable ideals and the family of all F_σ ideals, thereby indicating a possible direction for future research.

The final chapter is based on results that were unpublished at the time of the dissertation submission. It concerns microscopic sets studied using the language of game theory, as well as mapping porous sets via the classical mapping between the Cantor space and the unit interval, known as the Cantor-Lebesgue function. In the first part, we construct a Cantor-space counterpart of the σ -ideal $\mathcal{M}_{(f_n)}^*$, consisting of all sets contained in an F_σ set belonging to the family of (f_n) -microscopic sets $\mathcal{M}_{(f_n)}$. This construction leads to the definition of an appropriate game, which provides a convenient tool for studying the relationship between microscopic and porous sets. In the second part, we study the mapping of porous sets via the aforementioned mapping, building on the results of [14], where the authors investigate the mapping of microscopic sets. We prove that this mapping establishes an isomorphism between the families of lower porous sets on the Cantor space and the unit interval. In the case of upper porosity, the situation is more subtle – we obtain implications in both directions, one of which requires an additional assumption.