UNIVERSIDAD DE LA FRONTERA

REPORT ON PH.D. THESIS DISSERTATION FOR UNIVERSITY OF GDAŃSK:
TITLE: CONNECTED COMPONENTS OF THE SINGULAR LOCUS OF THE MODULI SPACE OF COMPACT RIEMANN SURFACES
AUTHOR: JAKUB SZMELTER-TOMCZUK
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To: Gdańsk University.

Below is my report on the Ph.D. dissertation titled *Connected components of the singular locus of the moduli space of compact Riemann surfaces* by Jakub Szmelter-Tomczuk.

2. Preliminaries

Let X be a closed orientable surface of genus $g \geq 2$.

2.1. **The Teichmüller space.** A marking for X is a pair (f, S), where S is a closed Riemann surface of genus g and $f : X \to S$ is an orientation-preserving homeomorphism.

Two markings (f_1, S_1) and (f_2, S_2) are equivalent if there is a biholomorphism $\phi : S_1 \to S_2$ such that $f_2^{-1} \circ \phi \circ f_1 : X \to X$ is isotopic to the identity.

The set of these equivalence classes is the Teichmüller space $\mathcal{T}_g$ of X; a contractible complex manifold of dimension $3(g-1)$.

2.2. **The modular group and the moduli space.** Let $\mathrm{Hom}^+(X)$ be the group of orientation-preserving homeomorphisms of X (with the compact open topology is a topological group). Let $\mathrm{Hom}_0(X)$ be the normal subgroup of $\mathrm{Hom}^+(X)$ consisting of those homeomorphisms that are isotopic to the identity (a closed subgroup). The quotient group $\mathrm{MCG}(X) = \mathrm{Hom}^+(X)/\mathrm{Hom}_0(X)$ (which is a discrete topological group with the induced topology from that of $\mathrm{Hom}^+(X)$) is the modular group of X.

If $[(f, s)] \in \mathcal{T}_g$, then the $\mathrm{MCG}(X)$-stabilizer of $[(f, S)]$ corresponds to the group $\mathrm{Aut}(S)$ of conformal automorphisms of S, which is known to be a finite group.

The modular group $\mathrm{MCG}(X)$ is known to act properly discontinuously as a group of holomorphic automorphisms of $\mathcal{T}_g$. This asserts that the quotient space $\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}(X)$, called the moduli space of X, has the structure of a complex orbifold of dimension $3(g-1)$.

Let $\pi : \mathcal{T}_g \to \mathcal{M}_g : [(f, S)] \mapsto [S]$ be the branched regular covering map with $\mathrm{Deck}(\pi) = \mathrm{MCG}(X)$.

The branch locus $\mathcal{B}_g \subset \mathcal{M}_g$ of π consists of the (biholomorphic) classes $[S]$ of Riemann surfaces structures S on X for which $\mathrm{Aut}(S)$ is non-trivial.

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2.3. **The singular locus.** The singular locus $\mathcal{S}_g \subset \mathcal{B}_g$ is the locus where $\mathcal{M}_g$ fails to be a topological manifold.

If $g = 2$, then $\mathcal{S}_2$ consists of one point.

If $g = 3$, then $\mathcal{S}_3$ consists of those $[S]$ such that S admits nontrivial automorphisms other than hyperelliptic involution.

If $g \geq 4$, then $\mathcal{B}_g = \mathcal{S}_g$.

2.4. **The algebraic structure.** It is also known that the moduli space $\mathcal{M}_g$ carries the structure of a quasi-algebraic complex variety of dimension $3(g-1)$, so one may also consider the connectivity (and connected components) of $\mathcal{S}_g$ in the corresponding Zariski topology.

In the literature, these two perspectives have created some confusion. One of the main results of the thesis is the fact that connected components of $\mathcal{S}_g$ in both topologies are the same. This result clarifies the previous discussions.

3. Main Thesis problem: Connectivity of the singular locus

The main subject of study in the reviewed thesis is about the connectivity and the description of the connected components of the singular locus $\mathcal{S}_g$.

3.1. **Broughton's stratum.** To study the singular locus, the author relies on Broughton's strata associated with finite group actions on X.

Let G be a finite group and assume there is a faithful homomorphism $\theta : G \to \mathrm{MCG}(X)$ (by Nielsen's realization theorem, this homomorphism lifts to a faithful homomorphism $\rho : G \to \mathrm{Hom}^+(X)$).

Let $\mathrm{Fix}(\theta(G))$ be the locus of those points $[(f,S)] \in \mathcal{T}_g$ such that $\theta(T)([(f,S)]) = [(f,S)]$, for every $T \in G$. In particular, $\theta(G) \leq \mathrm{Aut}(S)$, for $[(f,S)] \in \mathrm{Fix}(\theta(G))$. This turns out to be a smooth connected complex submanifold of $\mathcal{T}_g$ (isomorphic to the Teichmüller space of the quotient orbifold $S/\theta(G)$. The sub-locus $\mathrm{Fix}^0(\theta(G)) \subset \mathrm{Fix}(\theta(G))$ consists of those points $[(f,S)]$ such that $\theta(G) = \mathrm{Aut}(S)$.

The stratum associated to θ is the set $\mathcal{M}_g^{G,\theta} = \pi(\mathrm{Fix}^0(\theta(G)))$. This is (if non-empty) a connected and smooth complex submanifold of $\mathcal{M}_g$.

Its closure is $\overline{\mathcal{M}_g^{G,\theta}} = \pi(\mathrm{Fix}(\theta(G)))$ is a closed, connected, and irreducible complex algebraic subvariety of $\mathcal{M}_g$ (this is the union of some finite number of strata).

If $H \leq G$, then $\mathcal{M}_g^{G,\theta} \subset \overline{\mathcal{M}_g^{H,\theta}}$. Also, if $\theta_1(G)$ and $\theta_2(G)$ are conjugated in $\mathrm{MCG}(X)$, then $\mathcal{M}_g^{G,\theta_1} = \mathcal{M}_g^{G,\theta_2}$.

3.2. **$\mathcal{S}_g$ as a union of strata.** By the definition of the singular locus, and since for $g \geq 2$ there are a finite number of finite groups (up to conjugation) of $\mathrm{MCG}(X)$, one observes that $\mathcal{S}_g$ is a finite union of strata. As every finite group contains as a subgroup a cyclic group of prime order, each stratum is contained in the closure of the stratum associated with a cyclic group of prime order. In this way, $\mathcal{S}_g$ is the finite union of the closure of non-empty strata of cyclic groups of prime order.

As a consequence, each connected component of $\mathcal{S}_g$ is composed of a finite union of the closure of non-empty strata of cyclic groups of prime order.

3.3. **An special component of $\mathcal{S}_g$.** For every $g \geq 2$, there is an special connected component $C_g^{2,3}$ of $\mathcal{S}_g$. This component contains all strata associated with cyclic groups of orders 2 and 3. In this thesis, all other connected components are referred to as ordinary components.

Some partial study on the number of strata contained in $C_g^{2,3}$ is obtained (this is done by studying a certain graph whose vertices are the different non-empty strata for fixed g and edges between vertices if the closures of the corresponding strata intersect).

3.4. **Ordinary components.** One of the main results of this thesis is the following: if C is an ordinary component, then there is a prime integer $p \geq 5$, and a faithful homomorphism $\theta : \mathbb{Z}_p \to \mathrm{MCG}(X)$, such that $C = \overline{\mathcal{M}_g^{\mathbb{Z}_p,\theta}}$.

The ordinary component C could consist of one stratum (so such a stratum is closed), or it might contain at least two strata (each of them corresponding to certain cyclic p-groups).

Another of the results of the thesis is: if $g \leq 9$, then each ordinary connected component is a closed stratum of a cyclic group of prime order.

Also, it is studied those prime order cyclic group actions producing such a closed stratum (which provides isolated ordinary components). These actions are called maximal, in the sense that (by looking at them inside $\mathrm{MCG}(X)$) they cannot be a proper subgroup of a finite subgroup of $\mathrm{MCG}(X)$.

For $g = 10$, it is construct an ordinary component containing at least two different strata. In fact, for each prime $p \geq 5$ and each $n \geq 2$, it is observed that there are values of g such that $\mathcal{S}_g$ has an ordinary component containing at least n strata associated to p-groups.

3.5. **On the connectivity of $\mathcal{S}_g$.** In previous works of Bartolini, Costa, and Izquierdo, it was observed that $\mathcal{S}_g$ is connected for $g \in \{3, 4, 7, 13, 17, 19, 59\}$ and claimed that for the complementary genera it is non-connected. In this thesis, it was observed that $\mathcal{S}_{149}$ is also connected (proving that the above claim is incorrect as stated). Moreover, it is shown that the connectivity only holds for finitely many values of g. Finally, it is conjectured that the only cases for connectivity of $\mathcal{S}_g$ are for $g \in \{3, 4, 7, 13, 17, 19, 59, 149\}$.

Summary Report

The thesis under review studies the singular locus. It is proved that the connected components in both natural topologies on $\mathcal{M}_g$ coincide. This provides a clarification on this subject in both situations. Secondly, it is observed that the connected components of the singular locus, apart from the special one $C_g^{2,3}$, are given as the closure of the stratum of a prime order cyclic group. There are also described those ordinary components which are one stratum of a prime order cyclic group (by studying maximal cyclic actions). It has corrected a result on the connectivity of $\mathcal{S}_{149}$.

Some of the results of the thesis have already been published in good journals, including one in the Journal of Algebra, which is a highly respected journal. All of the above allows me to see that this is an outstanding piece of work. So, I propose that the dissertation be awarded a distinction.

Some comments/details

(1) Page 9, Line 2: Perhaps to replace "Fuchsian group" by "co-compact Fuchsian group". A Fuchsian group is just a discrete subgroup and it might be non-compact.
(2) Page 9, after the area formula: change "are" by "area".
(3) Page 12, part (6): It seems that case (e) is not possible. This will assert the existence of a closed Riemann surface of genus 2 admitting an automorphism of order three with a unique fixed point.

(4) Page 22: In the proof of Lemma 2.3., the situation $t + u \neq 1$ (line 15) is a consequence of the fact that the order q automorphism (on the underlying Riemann surface of $S/\mathbb{Z}_p$) cannot have only one fixed point. Also, in the same proof, some lines below, it seems that $n = h_2$?

(5) Page 25, just before Corollary 2.8: Perhaps to say (?) "The following corollary is a prompt consequence of the above Theorem, Lemma 1.10. and Theorem 2.5 "

(6) Page 47, Example 4.8: See Figure below

The above construction may seem a bit intricate so we present an example to illustrate our proof.

Example 4.8. Consider a signature $(10, 31, 31, 31)$. We will construct a finitely maximal action of Z_{31} with this signature following our construction. There are two prime ordered subgroups of $\mathbb{Z}_{31}^*$ whose orders divide the number of fixed points i.e. the subgroups of orders 2 and 3. Let $S = \{5, 30\}$ be a set of generators of these groups. Then the set $\mathbb{Z}_{31}^* \setminus \{1, 2, 3, 6, -6, -1, -2, -3, 0\}$ can be decomposed into sets $S_1 = \{5\}$, $-S_1 = \{26\}$ and $B = \{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27\}$. So, we take the following

$$n_1 = 30, n_2 = 29, n_3 = 28, n_4 = 6, n_5 = 5, n_6 = 26.$$

Now, we need to choose eighteen elements from the set B. However, it is clear that we may simply take suitable pairs $x, -x$ from the set B.

FIGURE 1

Rubén A. Hidalgo
Departamento de Matemática y Estadística,
Universidad de La Frontera, Temuco – Chile
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