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Report on the Ph. D. dissertation of Jakub Szmelter-Tomczuk

The doctoral dissertation of Jakub Szmelter-Tomczuk is about the singular set of the moduli space of Riemann surfaces. Let me briefly sketch the general context.

Context.

On a fixed oriented surface Σ_g of genus g the set of all complex structures (or, equivalently, hyperbolic structures) is unacceptably large. However, if we identify isomorphic structures, we obtain a finite dimensional space, the moduli space $\mathcal{M}_g$. There is an intermediate step—one can use only isomorphisms that are homotopic to identity; then the quotient space is called the Teichmüller space, $\mathcal{T}_g$, and is homeomorphic to $\mathbf{R}^{6g-6}$. The moduli space $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ under the action of a discrete group, the so-called mapping class group. This group action is free on a dense open subset of $\mathcal{T}_g$, but does have points with non-trivial stabilisers: symmetric complex/hyperbolic structures—that lead to singular points in $\mathcal{M}_g$. The moduli space $\mathcal{M}_g$ is one of the fundamental objects of geometry, and should be studied in as much detail as possible. The investigations of Jakub Szmelter-Tomczuk form a substantial addition to this study.

Methods.

One might expect some heavy-duty algebro-geometric tools to play the main rôles in investigating a singular set of an algebraic variety. However, the description of the singular set of $\mathcal{M}_g$ in terms of symmetric hyperbolic surfaces offers a much more direct and elementary approach. If Σ_g (with some hyperbolic structure) has a (finite) symmetry group G, then the quotient $G\backslash\Sigma_g$ is a hyperbolic orbifold (a surface with conical singularities). This quotient has a *signature,* encoding the genus of the underlying topological surface and the types of all the conical singularities. Furthermore, the fundamental group of the orbifold $G\backslash\Sigma_g$ (a G-extension of $\pi_1(\Sigma_g)$) has a canonical presentation; the sequence of images of the generators of that presentation in the group G form a so-called *generating vector* (a sequence of elements of a finite group that generate and that satisfy one specific relation). These two notions—signature and generating vector—are the main technical tools reappearing throughout the thesis.

Let G be the (full) symmetry group of some hyperbolic structure on Σ_g. Then, the topological type of the action of G on Σ_g can usually be realised isometrically by many different hyperbolic structures on Σ_g—such structures form a locally closed subset of the singular set $\mathcal{S}_g$ of $\mathcal{M}_g$, called a (Broughton) stratum. To analyse $\mathcal{S}_g$ one seeks to understand how various strata fit together, ie which of them are in the closures of others. Group theoretically, this is a question about extending the G-action on Σ_g to an action of a larger finite group. The dissertation contains numerous statements, old and new, about extending a cyclic group action on Σ_g to a larger group; these statements provide a convenient toolbox which is extensively used to investigate relative positions of various strata in $\mathcal{S}_g$.

Even with these extension lemmas, a lot of work is done directly with Z_p-valued generating vectors, classifying or constructing sequences of mod-p residues with prescribed arithmetic and symmetry behaviour. The author meticulously checks many specific cases. I would describe the technology as standard; yet it is skillfully applied, and coupled with sheer computational power, leading to general and elegant results that go well beyond what has been known before.

A (welcome indeed) outlier to the above description appears in the last section. The author uses standard but advanced analytic numer theory (Linnik's bound on the smallest prime in an arithmetic progression) to show that $\mathcal{S}_g$ is not connected for sufficiently large g.

Results.

The thesis is packed with original, attractive statements. They are well described by the author is the "Outline..." section, so I will not be comprehensive.

The components of $\mathcal{S}_g$ are of two types: there is the exceptional one $\mathcal{C}_g^{2,3}$, where Z_2-action belong—and others, called ordinary.

In section 2, ordinary components are shown to be built from strata coming from cyclic p-group actions; examples with arbitrarily large number of strata are constructed.

In section 3, the structure of $\mathcal{C}_g^{2,3}$ is studied. The *Cornalba graph* crudely describes the structure of this component: it has one vertex per each Z_p-stratum in $\mathcal{C}_g^{2,3}$ (all primes p count); vertices corresponding to contiguous strata are joined by an edge. The author shows that the diameter of this graph is at most 5 for all g, and is at least 4 for infinitely many g.

In section 4, a complete classification of non-extendable (finitely maximal) Z_p-actions is established. It is also argued that such actions are in some sense prevalent among all Z_p-actions.

In section 5, the author proves that $\mathcal{S}_g$ is connected for finitely many g. He conjectures that this happens only for $g = 3, 4, 7, 13, 17, 19, 59$ and 149.

Write-up.

The dissertation is very well organised. It has a very informative introductory part, then a section gathering preliminaries. The main results are conveniently divided into 4 further sections, each with its own goals. The technical parts are difficult to read (of course), but the surrounding text is helpful and the ideas are clearly presented.

Some critique: on page 31 the author defines the graph $\mathcal{G}_g(p)$ and states that "a spanning tree of $\mathcal{G}_g(p)$ is a star of radius 1"; on the very next page he considers "vertices isolated in $\mathcal{G}_g(p)$". How come?

Conclusion.

In my opinion, the dissertation of Jakub Szmelter-Tomczuk is a fine piece of mathematical work. I recommend that the author be granted the doctoral degree.

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