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Connected components of the singular locus of the moduli space of compact Riemann surfaces.

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STRESZCZENIE ROZPRAWY DOKTORSKIEJ

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pt. "Connected components of the singular locus of the moduli space of compact Riemann surfaces."

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Przestrzeń modułów $\mathcal{M}_g$ zwartych powierzchni Riemanna rodzaju $g \geq 2$ będąc przestrzenią orbit naturalnego działania grupy klas odwzorowań $\mathfrak{M}_g$ na przestrzeni Teichmüllera $\mathcal{T}_g$, ma strukturę zespolonego orbifoldu wymiaru $3(g-1)$. Kanoniczne rzutowanie $\mathcal{T}_g \to \mathcal{M}_g$ jest nakryciem rozgałęzionym, którego zbiór rozgałęziony (ang. branch set) $\mathcal{S}_g$, nazywany obszarem osobliwym, jest złożony z punktów odpowiadających klasom izomorfizmów zwartych powierzchni Riemanna mających nietrywialne automorfizmy. Przedmiotem rozprawy doktorskiej są topologiczne własności $\mathcal{S}_g$, w szczególności spójność oraz struktura i własności składowych spójności obszaru osobliwego.

Rozprawa podzielona jest, wyłączając wstęp oraz opis wyników, na pięć rozdziałów.

Pierwszy rozdział poświęcony został na przedstawienie wstępnych informacji potrzebnych do zrozumienia dowodów zawartych w pozostałych rozdziałach, oraz na zaprezentowanie metodologii, której używamy.

Przedmiotem drugiego rozdziału rozprawy są tzw. zwyczajne składowe spójności (ang. ordinary connected components) obszaru osobliwego, tj. składowe, które nie zawierają punktów $\mathcal{M}_g$, odpowiadających powierzchniom mającym automorfizm rzędu 2. Jednym z najważniejszych wyników przedstawionych w tym rozdziale jest twierdzenie orzekające, iż każda zwyczajna składowa spójności jest składową Cornalby. Wnioskiem z tego jest fakt, że każda taka składowa jest nierozkładalną podrozmaitością algebraiczną przestrzeni modułów. Udowodniliśmy, że każda warstwa równosymetryczna (ang. equisymmetric stratum), zawarta w takiej składowej $\mathcal{C}$ jest zadana przez działanie cyklicznej p-grupy. Ponadto, wykazaliśmy, że liczba pierwsza p jest wyznaczona jednoznacznie dla $\mathcal{C}$, tzn. jeżeli dwie warstwy zadane przez działania p-grupy oraz q-grupy są zawarte w $\mathcal{C}$, to $p = q$. Udowodniliśmy również, że zwyczajne składowe spójności mogą składać się z dowolnie dużej liczby warstw, dla odpowiednio dużych g. Dla liczby pierwszej $p \geq 5$ oraz liczby całkowitej $n \geq 2$, skonstruowaliśmy zwyczajną składową spójności zawierającą przynajmniej n warstw zadanych przez działania p-grup. Jako wniosek wykazaliśmy, że najmniejsze g, dla którego istnieje zwyczajna składowa spójności zawierająca przynajmniej dwie warstwy, jest równe 10.

W trzecim rozdziale przedstawiamy wyniki dotyczące struktury tzw. niezwykłej składowej spójności $\mathcal{C}_g^{2,3}$ (ang. exceptional connected component), zawierającej wszystkie punkty $\mathcal{M}_g$ odpowiadające powierzchniom mającym automorfizm rzędu 2 lub 3. Wprowadzamy do badań tzw. graf Cornalby $\mathcal{G}_g$, którego wierzchołki są we wzajemnie jednoznacznej odpowiedniości ze składowymi Cornalby zawartymi w $\mathcal{S}_g$. Dwa wierzchołki są połączone

krawędzią, gdy odpowiadające im składowe mają niepustą część wspólną. Wykazujemy, że ograniczenie górne na średnicę tego grafu jest niezależne od g i jest równe 5 dla każdego rodzaju. Definiujemy zredukowany graf Cornalby $\widehat{\mathcal{G}_g}$ i wykazujemy, że jest on spójnym podgrafem $\mathcal{G}_g$, którego średnica również jest mniejsza niż 6. Ponadto wykazujemy, że średnica tego grafu jest równa 5 dla nieskończenie wielu g. Dla dowolnego podgrafu $\mathcal{H}$ grafu Cornalby wprowadzamy pojęcie grubej powłoki (ang. bold hull) $\overline{\mathcal{H}}$ i udowadniamy, że $\mathcal{G}_g = \overline{\overline{\mathcal{G}_g(2)}}$, gdzie $\mathcal{G}_g(2)$ jest podgrafem rozpiętym na wierzchołkach odpowiadających działaniom grupy Z_2. Wykazujemy, że średnica $\overline{\mathcal{G}_g(2)}$ jest równa 3 dla nieskończenie wielu g. Jako wniosek znajdujemy warunki konieczne na to, aby odległość między dwoma wierzchołkami $\mathcal{G}_g$ była równa 5.

Rozdział czwarty poświęcony został badaniu własności tzw. skończenie maksymalnych (ang. finitely maximal) działań Z_p. Warstwy odpowiadające tym działaniom są składowymi spójności $\mathcal{S}_g$, co czyni te działania dla nas szczególnie interesującymi. W pierwszym podrozdziale znajdujemy warunki konieczne oraz wystarczające na to, aby sygnatura $(h; p, \overset{r}{\ldots}, p)$ odpowiadała działaniu skończenie maksymalnemu. Jako rezultat sklasyfikowaliśmy te działania Z_p, których rozszerzalność jest w pełni zdeterminowana przez ich sygnaturę. W drugim podrozdziale wprowadzamy pojęcie kombinatorycznej gęstości podzbioru względem filtracji. Wykazujemy, że zbiór działań skończenie maksymalnych jest kombinatorycznie gęsty w zbiorze wszystkich działań grup Z_p o ustalonym wymiarze Teichmüllera, co oznacza, że w pewnym sensie działania skończenie maksymalne stanowią większość wszystkich działań Z_p.

W ostatnim rozdziale podejmujemy temat spójności obszaru osobliwego. W pracy [On the connectivity of branch loci of moduli spaces, *Ann. Acad. Sci. Fenn. Math.* **38** (1) (2013) 245–258.] G. Bartolini, A. F. Costa oraz M. Izquierdo wykazali, że $\mathcal{S}_g$ jest spójny wyłącznie dla $g = 3, 4, 7, 13, 17, 19, 59$. Usiłując znaleźć prostszy dowód istnienia składowej $\mathcal{C}_g^{2,3}$ przypadkiem znaleźliśmy błąd w dowodzie tego twierdzenia. Jak się okazuje błędu tego nie da się naprawić. Wykazujemy, że $\mathcal{S}_{149}$ jest również spójny. Obserwacja ta pozostawia możliwość, iż obszar osobliwy jest spójny dla nieskończenie wielu g. Używając klasycznych wyników z analitycznej teorii liczb wykazujemy, że tak nie jest. Stawiamy hipotezę, że $\mathcal{S}_g$ jest spójny wyłącznie dla $g = 3, 4, 7, 13, 17, 19, 59$ oraz 149.

SUMMARY OF DOCTORAL DISSERTATION

BY JAKUB SZMELTER-TOMCZUK

titled: "Connected components of the singular locus of the moduli space of compact Riemann surfaces."

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The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$, being the orbit space of the natural action of the mapping class group $\mathfrak{M}_g$ on the Teichmüller space $\mathcal{T}_g$ admits a structure of a complex orbifold of dimension $3(g-1)$. The canonical projection $\mathcal{T}_g \to \mathcal{M}_g$ is a branch covering whose branch set $\mathcal{S}_g$, called the singular locus, is composed of points corresponding to isomorphism classes of compact Riemann surfaces that admit nontrivial automorphisms. The subject of this dissertation is the connectedness and the structure and properties of connected components of the singular locus.

This dissertation is divided, excluding the introduction and the description of the results, into five sections.

The first section is devoted to presenting preliminary information needed for understanding the proofs presented in the remaining sections and to outlining the methodology that we use.

The subjects of the second section of the dissertation are so-called ordinary connected components of the singular locus, i.e., components that do not contain any point corresponding to a surface with an automorphism of order 2. One of the main results of this section is the theorem saying that every ordinary connected component is a Cornalba component. As a result, we showed that every such connected component is an irreducible algebraic subvariety of the moduli space. We further investigated the structure of ordinary connected components by proving that every equisymmetric stratum contained in such a component $\mathcal{C}$ is defined by an action of a cyclic p-group. Moreover, we showed that the prime number p is unique for $\mathcal{C}$, i.e., if two strata defined by actions of a p-group and a q-group are contained in $\mathcal{C}$, then $p = q$. We proved that ordinary connected components may be composed of arbitrarily many strata, for large enough genera. For a prime number $p \geq 5$ and an integer $n \geq 2$, we constructed an ordinary connected component composed of at least n strata defined by actions of p-groups. As a result, we proved that the minimal g for which an ordinary connected component composed of at least two strata exists is equal to 10.

In the third section, we present results concerning the structure of the exceptional connected component $\mathcal{C}_g^{2,3}$, which is the component containing points corresponding to surfaces admitting an automorphism of order 2 or 3. We introduce the Cornalba graph $\mathcal{G}_g$, whose vertices are in one-to-one correspondence with the Cornalba components contained in $\mathcal{S}_g$. Two vertices of $\mathcal{G}_g$ are connected by an edge if the intersection of the corresponding

components is nontrivial. We show that the upper bound on the diameter of this graph does not depend on g and equals 5 for every genus. We define the reduced Cornalba graph $\widehat{\mathcal{G}}_g$, and we prove that it is a connected subgraph of $\mathcal{G}_g$, whose diameter is also smaller than 6. Furthermore, we show that this diameter equals 5 for infinitely many genera. For a subgraph $\mathcal{H}$ of the Cornalba graph, we introduce the bold hull $\overline{\mathcal{H}}$, and we show that $\mathcal{G}_g = \overline{\overline{\mathcal{G}_g(2)}}$, where $\mathcal{G}_g(2)$ is the subgraph spanned on vertices corresponding to actions of Z_2. We prove that the diameter of $\overline{\mathcal{G}_g(2)}$ is equal to 3 for infinitely many genera. As a result, we find necessary conditions for two vertices to be distant by 5 in $\mathcal{G}_g$.

The fourth section is devoted to the study of so-called finitely maximal actions of Z_p. Equisymmetric strata defined by such actions coincide with certain connected components, which make them particularly interesting for us. In the first subsection, we find the necessary and sufficient conditions for a signature $(h; p, \overset{r}{\ldots}, p)$ to correspond to a finitely maximal action. As a result, we classify these actions of Z_p whose extendability is determined solely by its signature. In the second subsection, we introduce a notion of the combinatorial density of a subset with respect to a filtration. We show that the set of finitely maximal actions is combinatorially dense in the set of all actions of groups Z_p with a fixed Teichmüller dimension, which, in some sense, means that the vast majority of actions are finitely maximal.

In the last section, we consider the problem of the connectedness of the singular locus. In [On the connectivity of branch loci of moduli spaces, *Ann. Acad. Sci. Fenn. Math.* **38** (1) (2013) 245–258.] G. Bartolini, A. F. Costa, and M. Izquierdo showed that $\mathcal{S}_g$ is connected only for $g = 3, 4, 7, 13, 17, 19, 59$. While trying to find a simpler proof of the existence of $\mathcal{C}_g^{2,3}$, we identified a gap in the proof of this theorem. As it turns out, the flaw cannot be fully repaired. We show that $\mathcal{S}_{149}$ is also connected. This observation leaves the possibility that the singular locus is connected for infinitely many genera. Using some classical results from analytic number theory, we prove that this is not the case. We conjecture that $\mathcal{S}_g$ is connected only for $g = 3, 4, 7, 13, 17, 19, 59$ and 149.

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Introduction

The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ can be constructed in various ways. The most useful for our purposes is to view $\mathcal{M}_g$ as the quotient space of the Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group $\mathfrak{M}_g$. By the results of R. Fricke and F. Klein [17], and of O. Teichmüller himself [39], the Teichmüller space is homeomorphic to $\mathbb{C}^{3(g-1)}$. Therefore, $\mathcal{M}_g$ has the structure of a complex orbifold of dimension $3(g-1)$.

The canonical projection $\mathcal{T}_g \to \mathcal{M}_g$ is a branched covering with branch set $\mathcal{B}_g$, called the *branch locus*. For $g > 2$, this set consists of points corresponding to Riemann surfaces that have a nontrivial automorphism. Since every closed Riemann surface of genus 2 is hyperelliptic, $\mathcal{B}_2 = \mathcal{M}_2$. For this reason, $\mathcal{B}_2$ is usually restricted to points of $\mathcal{M}_2$ representing compact Riemann surfaces that admit nontrivial automorphisms other than hyperelliptic involution.

On the other hand, by the results of P. Deligne and D. Mumford [16], the moduli space is a Zariski-open subset of a projective variety $\overline{\mathcal{M}}_g$, called the *Deligne-Mumford compactification* of $\mathcal{M}_g$. As such, the moduli space is a quasi-projective variety whose set of singular points $\mathcal{S}_g$ is called the *singular locus*.

Every singular point of $\mathcal{M}_g$ is also a branch point, but the converse is not necessarily true. For genus 2, the singular locus consists of the unique point corresponding to the compact Riemann surface with an automorphism of order 5 [26]. The locus $\mathcal{S}_3$ consists of points corresponding to compact Riemann surfaces that admit nontrivial automorphisms other than hyperelliptic involution. For $g > 3$, the branch and singular loci coincide. Keeping these subtle differences in mind, we will abuse language and identify the branch locus with the singular locus.

Furthermore, the moduli space, as an orbifold, contains points having a neighborhood homeomorphic to a complex ball of dimension $3(g-1)$, and may contain points, called *topologically singular*, that do not have such a neighborhood. The existence of topologically singular points was proved by H. E. Rauch in [33]. Clearly, topologically singular points must be branch points; the converse does not need to be true. In [33] it was proved that for $g > 3$ every branch point is also a topologically singular point. Moreover, the branch loci $\mathcal{B}_2$ and $\mathcal{B}_3$ contain topologically singular points as well as nonsingular ones. The results of Rauch were later proved topologically by A. F. Costa and A. M. Porto in [15].

This thesis is devoted to the study of connected components of the singular locus. However, we need to clarify the topology in which we are working, since the moduli space admits at least two natural topologies: the one inherited from the Teichmüller space, which we will call the *Teichmüller topology* of $\mathcal{M}_g$, and the Zariski topology. In principle, these topologies do not need to coincide. A good example illustrating this issue, although

in the real affine case, is the hyperbola defined by the equation $xy = 1$. Clearly, this subset of $\mathbb{R}^2$ is not connected in the standard topology of the Euclidean plane, contrary to the Zariski topology of $\mathbb{R}^2$, in which the hyperbola is irreducible and thus connected. However, in [37] we show that the sets of connected components of $\mathcal{S}_g$ in both topologies coincide. Therefore, we do not need to restrict our attention to only one of them. It is worth mentioning that the connected components of $\mathcal{S}_g$ are also path-connected components in the Teichmüller topology, which gives our results a deeper meaning, since paths in $\mathcal{M}_g$ correspond to continuous deformations of compact Riemann surfaces.

One of the main tools we use is Broughton's equisymmetric stratification [11]. For a finite subgroup G of the mapping class group, the equisymmetric stratum $\mathcal{M}_g^G$ consists of points of $\mathcal{M}_g$ for which the full automorphism group of the corresponding compact Riemann surface induces a subgroup of $\mathfrak{M}_g$ conjugate to G. The Zariski closure $\overline{\mathcal{M}}_g^G$ of a nonempty stratum is obtained by dropping in its definition the assumption that the action of G is full. S. A. Broughton proved that nonempty strata are connected, smooth, and locally closed algebraic subvarieties of $\mathcal{M}_g$, while the Zariski closures of strata are irreducible algebraic subvarieties of the moduli space. These results can be seen as a generalization of the theorems of M. Cornalba from [13], where he observed that the singular locus is a union of irreducible subvarieties, which we shall call the *Cornalba components*, and the point is that these components are precisely the Zariski closures of nonempty strata defined by subgroups of $\mathfrak{M}_g$ of prime orders. In [37] we proved that Zariski closures of nonempty strata (Cornalba components in particular) are also connected in the Teichmüller topology of $\mathcal{M}_g$, which filled a gap that occurred in the study of $\mathcal{S}_g$ resulting from the simultaneous use of notions and objects from the Zariski and the Teichmüller topologies of the moduli space.

Using Broughton's stratification, G. Bartolini, A. F. Costa, and M. Izquierdo, occasionally supported by A. M. Porto, studied the connectedness of $\mathcal{S}_g$ in a series of papers [4], [7], and [8]. This research reached its culmination in [6], where it was proved that the singular loci are connected only for genera $3, 4, 7, 13, 17, 19, 59$. However, there is a gap in their proof, and unfortunately, the assertion itself is not correct. We will point out the flaw and attempt to correct it in Section 5.

The work [6] implies that $\mathcal{S}_g$ is disconnected for infinitely many genera, which motivates the study of the structure of its connected components. Bartolini and Izquierdo showed in [8] that all points corresponding to compact Riemann surfaces having an automorphism of order 2 or 3 are contained in the same connected component, which we denote by $\mathcal{C}_g^{2,3}$. For years, this component was the only one known to consist of more than one stratum. In [6], the authors asked whether there exist components other than $\mathcal{C}_g^{2,3}$ with this property. This question was answered in the affirmative by G. Gromadzki and the author of this thesis in [21]. It is important to mention here that the connected components constructed there coincide with certain Cornalba components.

These results imply that there are at least three natural types of connected components of singular loci: the components $\mathcal{C}_g^{2,3}$, which we call *exceptional*; components consisting of a single stratum; and components consisting of more than one stratum but coinciding with the Zariski closure of one of them. We will call components different from $\mathcal{C}_g^{2,3}$ *ordinary*.

Outline of the thesis and the description of results

We divided this thesis into five sections. In the first one, we present the preliminary results needed to understand and follow the proofs presented in the remaining parts, and we outline the methodology that we use.

In the second, we present results obtained in [37], which address a natural question prompted by the results of [21]: does there exist an ordinary connected component that does not coincide with a certain Cornalba component? We prove, in Theorem 2.6, that this is not the case. Furthermore, in Theorems 2.5 and 2.6, we prove that every stratum contained in an ordinary connected component $\mathcal{C}$ is defined by an action of a cyclic p-group. It is important to note that the prime number p is unique for $\mathcal{C}$. Consequently, if actions of a p-group and a q-group define strata contained in $\mathcal{C}$, then $p = q$. As a result, we find that every ordinary connected component of the singular locus is an irreducible algebraic subvariety of the moduli space, since it coincides with some Cornalba component.

We further study the structure of ordinary connected components by calculating signatures of actions of groups that define the strata contained in them. In particular, we show in Theorem 2.5 that if G is a p-group of order p^n, where $n > 1$, defining a stratum contained in an ordinary connected component, then the signature of its action equals $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$. This allowed us to precisely describe the form of generating vectors for actions of Z_p defining strata contained in ordinary connected components. In Theorem 2.10 we prove that a Cornalba component corresponding to an action of Z_p defined by a generating vector of given form is contained in $\mathcal{C}_g^{2,3}$ if and only if this action extends to an action of a dihedral group or Z_{pq} for some prime number $q \neq p$.

We show in Theorem 2.15 that ordinary connected components may be composed of arbitrarily many strata for large enough genera. Precisely, given a prime $p \geq 5$ and an integer $n \geq 2$, we construct an ordinary connected component, being the union of at least n strata defined by actions of p-groups. As a further consequence of this theorem, we get that the smallest value of g for which there exist ordinary connected components consisting of more than one stratum is equal to 10, which contradicts [6], where relying on [5] by Bartolini, the authors mentioned that for $g \leq 24$ there are no such components in $\mathcal{S}_g$.

The next section is devoted to studying combinatorial properties of the exceptional component. Since the singular locus $\mathcal{S}_g$ coincides with the union of all Cornalba components, its structure can be encoded in a combinatorial object known as the *singular nerve* which we shall shortly call *nerve*, denoted as $\mathcal{N}_g$. The concept of a nerve of open coverings of topological spaces was introduced by P. S. Alexandroff in [2], and its idea was later applied to the study of the singular locus by G. Gromadzki in [19].

The nerve $\mathcal{N}_g$ is an abstract simplicial complex whose vertices correspond to topological types of actions of cyclic groups Z_p on compact Riemann surfaces of genus g that define

nonempty strata in the moduli space. A finite set of vertices spans a simplex if and only if the corresponding Cornalba components have a nontrivial intersection, which in turn means that there exists a compact Riemann surface admitting automorphisms defining these components.

The connectedness of $\mathcal{N}_g$ is equivalent to the connectedness of the singular locus $\mathcal{S}_g$, and the structure of the connected components of the nerve reflects that of $\mathcal{S}_g$ itself. Moreover, the nerve introduces a range of combinatorial invariants into the study of the singular locus, such as geometric and homological dimensions, as well as the diameter of its 1-skeleton. Gromadzki studied the dimensions of $\mathcal{N}_g$ in [19], while in [23], jointly with A. Wootton investigated the structure of its 1-skeleton.

Since the Cornalba components are connected, there is a natural bijective correspondence between the connected components of $\mathcal{S}_g$ and the connected components of $\mathcal{N}_g$. Furthermore, ordinary connected components correspond to isolated points of the nerve, since, as we already mentioned, every such component coincides with some Cornalba component. The exceptional component $\mathcal{C}_g^{2,3}$ corresponds to the maximal connected subcomplex of $\mathcal{N}_g$. The 1-skeleton of this component is a connected graph which we will call the *Cornalba graph* and denote it by $\mathcal{G}_g$. In the third part of this thesis, we aim to investigate the structure of the exceptional component by examining the properties of this graph, paying special attention to its diameter. This problem proves to be highly technical and challenging to address. However, we were able to obtain interesting and strong results, but not fully satisfactory. First, in Theorem 3.4 we prove that the upper bound on the diameter of the Cornalba graph does not depend on g and, in particular, is smaller than 6 for all genera. We do this by explicitly constructing a path between two arbitrary vertices of length at most 5. Furthermore, in Theorem 3.8 we prove that for infinitely many genera, the diameter of the Cornalba graph is greater than 3; however, we cannot point to an infinite sequence of genera for which the diameter equals 3. All of this gives rise to natural questions and conjectures, which we do not dare to pose definitively at this stage of our knowledge.

Let v and w be two vertices of $\mathcal{G}_g$ corresponding to actions of Z_p and Z_q, respectively, connected by an edge e. Let G be an automorphism group of a compact Riemann surface of genus g of the smallest order that extends these actions. By a *weight* of e we mean the number $w(e) := |G|/pq$. We define a subgraph $\widehat{\mathcal{G}}_g$ called the *reduced Cornalba graph* consisting of all vertices of $\mathcal{G}_g$ but only those edges e for which $w(e) = 1$, which we shall call *bold*. We observe, using Lemma 3.3 and Theorem 3.4, that the reduced Cornalba graph is connected and that its diameter cannot exceed 5. Finally, in Theorem 3.13 we prove that its diameter is equal to 5 for infinitely many genera.

For a subgraph $\mathcal{H}$ of the Cornalba graph, we introduce a notion of the *bold hull* $\overline{\mathcal{H}}$ of $\mathcal{H}$ i.e. the subgraph spanned by all vertices of $\mathcal{H}$ and vertices connected to some vertex of $\mathcal{H}$ by bold edges. We show that $\mathcal{G}_g = \overline{\overline{\mathcal{G}_g(2)}}$, where $\mathcal{G}_g(2)$ is the subgraph spanned on

vertices corresponding to actions of Z_2. Moreover, we study the diameter of $\overline{\mathcal{G}_g(2)}$ and in Proposition 3.10 show that it equals 3 for infinitely many genera. As a result, we obtain necessary conditions for two vertices to be distant by 5 in the Cornalba graph.

The fourth section is devoted to the study of the so-called *finitely maximal* actions of Z_p on compact Riemann surfaces, which are those that when viewed as the subgroups of $\mathfrak{M}_g$ have no finite extensions there. The strata corresponding to such actions form connected components of singular loci, which makes them particularly interesting; consequently, these actions were extensively studied, e.g. [14], [18], [20] [32], [36].

In the first subsection of this part, we build upon the work of V. Peterson, J. Russell, and A. Wootton [32], who established sufficient conditions for the existence of a finitely maximal action of Z_p with the signature $(h; p, .^r., p)$. Since their conditions are not necessary, the authors noted that a deeper investigation into which signatures correspond to finitely maximal actions would be a valuable direction for future research. In Theorem 4.3, we determine the necessary and sufficient conditions for such a signature to correspond to a finitely maximal action of Z_p, thereby providing a complete classification of such signatures. Perhaps more intriguingly, this classification also allows us to identify precisely which signatures do not correspond to finitely maximal actions. As a consequence, we can now pinpoint the cases in which the extendability of an action of Z_p is completely determined only by its branching data, which directly address the remarks made at the end of [32].

The second subsection concern results of [20] by G. Gromadzki and the author. The results of Petersson, Russell, and Wootton from [32] (cf. Theorem 5) could indicate that the number of finitely maximal actions of Z_p on a surface of genus g is much smaller than the number of extendable actions, since every action with nonminimal orbit genus is extendable. This assertion could also be indicated by a series of papers which main goal was to establish genera for which maximal actions exist at all, cf. [6], [8], [14], [35]. On the other hand, generating vectors of extendable actions must obey certain rules given by Lemmas 1.9-1.12, which may suggest that the majority of actions of Z_p with the signature $(h; p, .^r., p)$ are finitely maximal, provided at least one maximal action with such a signature exists. To explain what is going on, we introduce a notion of the *combinatorial density* of a subset of a countable set with respect to a filtration of this set. We show that the set of finitely maximal actions is combinatorially dense in the set of all topological equivalence classes of actions of groups of prime orders having a fixed Teichmüller dimension with respect to a certain natural filtration of the latter. This means that the majority of actions of Z_p are finitely maximal in some sense, which may be suprising in the light of the observation made at the beginning.

As it turns out, the collection of classes of prime ordered actions admits an additional natural filtration indexed by genera . Motivated by intuitions arising from the results of [32], we are fairly confident that, with respect to this filtration, the set of *extendable*

actions is combinatorially dense in the set of all action classes. Exploring these phenomena appears to be a promising and interesting direction for future research.

The last section concerns results obtained in [38]. Our initial goal here was to simplify the proof of the existence of the exceptional connected component [8] based on our observation from [37] on the connectivity of Cornalba components in the topology inherited from the Teichmüller space. However, while pursuing this goal, we accidentally discovered a gap in the proof of the main theorem of [6]. As it turns out, the flaw cannot be fully repaired, and the list of genera for which the singular loci are connected given in [6], turned out to be not complete. In particular, we show in Theorem 5.9 that $\mathcal{S}_{149}$ is also connected. Our methods, used here can be extended to other cases, resulting also in a simpler way of proving that, $\mathcal{S}_g$ for $g = 3, 4, 7, 13, 17, 19$, and 59 are also connected. Theorem 5.9 leaves open the possibility that there are infinitely many genera with this property. Using some results from analytic number theory, we prove, in Theorem 5.11, that this is not the case. Moreover, we conjecture that there are no such genera other than those listed above.

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1. Preliminaries

1.1. **Compact Riemann surfaces as orbit spaces.** A subgroup Λ of the group of orientation-preserving isometries of the hyperbolic plane is called a Fuchsian group if and only if it is discrete and the surface $\mathcal{H}/\Lambda$ is compact. Every Fuchsian group Λ is finitely presented and by the results of H. C. Wilkie [40], has the following presentation

$$\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r \mid \gamma_1^{m_1}, \ldots, \gamma_r^{m_r}, \gamma_1 \ldots \gamma_r [\alpha_1, \beta_1] \ldots [\alpha_h, \beta_h] \rangle \tag{1}$$

which we shall call *canonical*. The number h is called the *orbit genus* and is equal to the genus of the surface $\mathcal{H}/\Lambda$. Elements α_i, β_i are called *hyperbolic* and represent hyperbolic translation. The numbers m_i are called *periods*. Generators γ_i are called *elliptic* and represent rotations of the hyperbolic plane. Every element of Λ of finite order is conjugate to a power of some elliptic generator. The canonical presentation (1) of a Fuchsian group Λ can be codified into a sequence of numbers

$$(h; m_1, \ldots, m_r) \tag{2}$$

called its *signature*. A Fuchsian group Γ without elliptic generators is called a *surface Fuchsian group*. To a Fuchsian group Λ with the signature (2) we can associate a number

$$\mu(\Lambda) = 2(h-1) + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}$$

which we will call the *are* of Λ. An abstract group Λ with presentation (1) is isomorphic to a Fuchsian group if and only if $\mu(\Lambda) > 0$.

By the Riemann uniformization theorem, every compact Riemann surface S of genus $g > 1$ is isomorphic to an orbit space $\mathcal{H}/\Gamma$ of an action of some surface Fuchsian group. Furthermore, there exists a Fuchsian group Λ containing Γ as a normal subgroup such that $\mathrm{Aut}(S) \cong \Lambda/\Gamma$.

Conversely, if $\theta\colon \Lambda \to G$ is a torsion-free kernel epimorphism, called a *smooth epimorphism*, from a Fuchsian group Λ onto a finite group G, then $\Gamma = \ker\theta$ is a surface Fuchsian group and G is isomorphic to an automorphism group of $S = \mathcal{H}/\Gamma$. We will say that θ *defines an action* of G on S, and if Λ has the signature (2), then we will say that G *acts with the signature* (2). The genus of S can be found using the Riemann-Hurwitz formula, which states that if a Fuchsian group Δ is a subgroup of a Fuchsian group Δ', then

$$[\Delta' : \Delta] = \frac{\mu(\Delta)}{\mu(\Delta')}.$$

The following is an immediate consequence of the Riemann-Hurwitz formula.

Lemma 1.1. *Suppose that Z_p acts on a compact Riemann surface of genus g with the signature $(h; p, \overset{r}{\ldots}, p)$. Then, $(p-1)/2$ divides $g - h$.*

Proof. By the Riemann-Hurwitz formula $2(g-1) = 2p(h-1) + r(p-1)$ which gives $g - h = (r + 2(h-1))(p-1)/2$ and concludes the proof. □

At the end of this subsection, we want to mention a particular form of the well-known Castelnuovo-Severi inequality, proof of which can be found in [1].

Lemma 1.2. *Let S be a compact Riemann surface of genus g admitting two distinct automorphism groups H_1, H_2 of prime orders p_1, p_2 respectively. Then,*

$$g \leq p_1 h_1 + p_2 h_2 + (p_1 - 1)(p_2 - 1), \tag{3}$$

where h_i is the genus of the surface S/H_i. □

1.2. **Topological classification of conformal finite group actions on compact surfaces.** Let $\theta_1 \colon \Lambda_1 \to G_1$ and $\theta_2 \colon \Lambda_2 \to G_2$ be smooth epimorphisms defining actions of groups G_1 and G_2 on compact Riemann surfaces of genus g. We say that these actions are *topologically equivalent* if and only if the diagram

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\varphi} & \Lambda_2 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ G_1 & \xrightarrow{\psi} & G_2 \end{array}$$

commutes for some isomorphisms φ and ψ. If we assume φ and ψ to be monomorphisms, then we say that the action of G_2 *extends the action of* G_1. We will say that an action of a group G is *finitely maximal* if it does not extend to an action of any proper supergroup of G.

A sequence $(a_1, b_1, \ldots, a_h, b_h; x_1, \ldots, x_r)$ of elements of a finite group G is called a $(h; m_1, \ldots, m_r)$-*generating vector*, or simply a *generating vector* if it does not bring any confusion, of an action of G on a compact Riemann surface of genus g if and only if

- its elements generate G,
- the order of x_i is equal to m_i,
- $x_1 \ldots x_r [a_1, b_1] \ldots [a_h, b_h] = 1$,
- $2(g-1) = |G| \left(2(h-1) + \sum_{i=1}^{r} (1 - 1/m_i)\right)$.

Observe that for a smooth epimorphism $\theta \colon \Lambda \to G$, the sequence of images of a canonical set of generators of Λ forms a generating vector for the action of G defined by θ. On the other hand, having a $(h; m_1, \ldots, m_r)$-generating vector for G we can define a smooth epimorphism θ from a Fuchsian group Λ with the signature (2) onto G. We get the following theorem due to the Nielsen classification [31]

Theorem 1.3. *Let*

$$(a_1, b_1, \ldots, a_h, b_h; x_1, \ldots, x_r) \quad \text{and} \quad (a'_1, b'_1, \ldots, a'_h, b'_h; x'_1, \ldots, x'_r),$$

be two generating vectors for Z_p. These vectors define topologically equivalent actions of Z_p if and only if there exist a permutation $\sigma \in S_r$ and an automorphism φ of Z_p such that

$$(\varphi(x_{\sigma(1)}), \ldots, \varphi(x_{\sigma(r)})) = (x'_1, \ldots, x'_r).$$

□

1.3. **Extensions of actions of groups of prime orders.** We start this subsection by presenting one of the main tools used in this dissertation, the Theorem of MacBeath proved in [30].

Lemma 1.4. *Let $S = \mathcal{H}/\Gamma$ be a compact Riemann surface with an automorphism group $G \cong \Lambda/\Gamma$ and let $\gamma_1, \ldots, \gamma_r$ be a set of elliptic canonical generators of Λ whose periods are $m_1, \ldots, m_r$ respectively. Let $\theta\colon \Lambda \to G$ be the canonical projection. Then, the number $F(\varphi)$ of points of S fixed by a nontrivial element φ of G is given by the formula*

$$|N_G(\langle\varphi\rangle)| \sum_i \frac{1}{m_i},$$

where the operator N stands for the normalizer and the sum is taken over those i for which φ is conjugate to a power of $\theta(\gamma_i)$. □

By the above lemma the number of fixed points of an automorphism φ of prime order is equal to the number of periods in the signature of the action of $\langle\varphi\rangle$. Therefore, the above lemma, together with the Riemann-Hurwitz formula, can be used to find this signature. We will use it mainly for this purpose. The following useful lemmas can be found in [32].

Lemma 1.5. *Suppose that an action of Z_p on a compact Riemann surface with the signature $(h; p, \overset{r}{\ldots}, p)$, where $r \geq 3$, extends to an action of a proper supergroup G, and assume that the normalizer of Z_p in G is equal to Z_p. Then, $h \geq (p-3)/2$.* □

Lemma 1.6. *Suppose that Z_p acts on a compact Riemann surface S with the signature $(h; p, \overset{r}{\ldots}, p)$, where $r \geq 3$, $h < (p-3)/2$ and assume that Z_p is a proper subgroup of $\mathrm{Aut}(S)$. Then, Z_p is contained as a normal subgroup of prime index q in some subgroup K of $\mathrm{Aut}(S)$.*

Lemma 1.7. *Suppose that Z_p acts with the signature $(h; p, \overset{r}{\ldots}, p)$ on a compact Riemann surface S of genus $g \geq 2$, such that $r \leq 2$ or $r \geq 3$ and $h \geq (p-3)/2$. Then, this action extends to an action of a group of order $2p$.* □

The following is an easy corollary from Lemma 1.5 and the above Lemma.

Corollary 1.8. *Let G be an automorphism group of a compact Riemann surface. Suppose that a subgroup $Z_p < G$ coincides with its normalizer in G. Then, the induced action of Z_p extends to an action of a group of even order.* □

The following corollary from the Riemann-Hurwitz formula, and Lemma 1.4 will be particularly useful for our purposes.

Lemma 1.9. *Let G be a group of order pq for some prime numbers p, q acting on a compact Riemann surface S of genus $g \geq 2$, and let Z_p be a normal subgroup of G acting with the signature $(h; p, \overset{r}{\ldots}, p)$. Then, G acts with the signature*

$$(k; q, \overset{s}{\dots}, q, p, \overset{t}{\dots}, p, pq, \overset{u}{\dots}, pq). \tag{4}$$

for some $k, s, t, u \geq 0$*, where*

$$r = tq + u,$$

$$2(h-1) - 2q(k-1) = (s+u)(q-1). \tag{5}$$

Moreover, if $h \leq 2$*, then*

$$\begin{array}{ll} \text{(a) } h=2, k=0, s+u=6, q=2 & \text{(b) } h=2, k=0, s+u=4, q=3 \\ \text{(c) } h=2, k=0, s+u=3, q=5 & \text{(d) } h=2, k=1, s+u=2, q=2 \\ \text{(e) } h=2, k=1, s+u=1, q=3 & \text{(f) } h=1, k=0, s+u=q=3 \\ \text{(g) } h=1, k=0, s+u=4, q=2 & \text{(h) } h=1, k=1, s+u=0, \ q \text{ arbitrary} \\ \text{(i) } h=0, k=0, s+u=2, \ q \text{ arbitrary.} & \end{array} \tag{6}$$

Furthermore, $u \leq 2h+2$ *and if in addition* $q > 2$ *then* $u \leq h+2$*.*

Proof. Clearly, the only periods in the signature of the action of G can be p, q, and pq and so we get (4). By Lemma 1.4 $r = qt + u$, and by the Riemann-Hurwitz formula we get (5). This easily gives us (6). Using (5) again we get that

$$(s+u)(q-1) = 2(h-1) - 2q(k-1) \leq 2(h-1) + 2q = 2h + 2(q-1)$$

thus

$$s + u \leq 2\frac{h}{q-1} + 2$$

which easily gives that $u \leq 2h+2$ for all q, and $u \leq h+2$ when $q > 2$. □

In general settings, finding a generating vector of an induced action of a subgroup H of a group G acting on a compact Riemann surface is not an easy task. However, in some special cases, it is possible with a little effort. The following three lemmas, which can be found in [32] (cf. also [41]), provide explicit forms for the generating vectors of actions induced by groups of order pq for primes p and q.

Lemma 1.10. *Let* $(a_1, b_1, \dots a_k, b_k, x_1, \dots, x_s, y_1, \dots, y_t, z_1, \dots, z_u)$*, where* $s + u > 0$ *and* $t = 0$ *if* $p = q$*, be a* $(k; p, \overset{s}{\dots}, p, q, \overset{t}{\dots}, q, pq, \overset{u}{\dots}, pq)$*-generating vector defining an action of* Z_{pq}*. Then the corresponding* $(h; p, \overset{r}{\dots}, p)$*-generating vector for* $Z_p \trianglelefteq Z_{pq}$ *is equivalent to*

$$(1, \overset{2h}{\dots}, 1, x_1, \overset{q}{\dots}, x_1, \dots, x_s, \overset{q}{\dots}, x_s, z_1^q, \dots, z_u^q),$$

where h *and* r *can be found using the Riemann-Hurwitz formula and Lemma* 1.4*.* □

Lemma 1.11. *Let* $(a_1, b_1, \dots, a_k, b_k, x_1, \dots, x_s, y_1, \dots, y_t)$*, where* $s + t > 0$*, be a* $(k; p, \overset{s}{\dots}, p, q, \overset{t}{\dots}, q)$*-generating vector defining an action of* $Z_p \rtimes Z_q$ *and let* λ *be a* q*-th root of*

the unity modulo p i.e $\lambda^q = 1 \pmod p$. Then the corresponding $(h; p, \overset{r}{\ldots}, p)$-generating vector for $Z_p \trianglelefteq Z_p \rtimes Z_q$ is equivalent to

$$(1, \overset{2h}{\ldots}, 1, x_1, x_1^{\lambda}, \ldots, x_1^{\lambda^{q-1}}, \ldots, x_s, x_s^{\lambda}, \ldots, x_s^{\lambda^{q-1}}),$$

where h and r can be found using the Riemann-Hurwitz formula and Lemma 1.4. □

Lemma 1.12. *Let $(a_1, b_1, \ldots, a_k, b_k, x_1, \ldots, x_s, y_1, \ldots, y_t)$, where $s + t > 0$, be a $(k; p, \overset{s}{\ldots}, p, p, \overset{t}{\ldots}, p)$-generating vector defining an action of $Z_p \times Z_p$, such that $x_1, \ldots, x_s \in Z_p$ and $y_1, \ldots, y_t \notin Z_p$. Then the corresponding $(h; p, \overset{r}{\ldots}, p)$-generating vector for $Z_p \trianglelefteq Z_p \times Z_p$ is equivalent to*

$$(1, \overset{2h}{\ldots}, 1, x_1, \overset{p}{\ldots}, x_1, \ldots, x_s, \overset{p}{\ldots}, x_s),$$

where h and r can be found using the Riemann-Hurwitz formula and Lemma 1.4. □

Throughout the rest of the thesis, by a nontrivial action of Z_q on Z_p, we shall understand an action defined by a nontrivial homomorphism $Z_q \to \mathrm{Aut}(Z_p)$. The following is an easy corollary of the above three lemmas and Lemma 1.9.

Corollary 1.13. *Let $(1, \overset{2h}{\ldots}, 1; a_1, \ldots, a_r)$ be a generating vector for an action of Z_p, such that $r \geq 2h+3$, elements a_i are pairwise different and it cannot be decomposed into orbits of a nontrivial action of any Z_q on Z_p. Then, the action of Z_p is finitely maximal.* □

1.4. **Generalizations and further results on the extendability of actions of cyclic groups.** In the forthcoming subsection, we present some stronger results concerning the extendability of actions of cyclic groups, some of which are original and have not yet been published.

First, observe that a $(h; p, \overset{r}{\ldots}, p)$-generating vector

$$\tau = (1, \overset{2h}{\ldots}, 1; a_1, \overset{r}{\ldots}, a_r),$$

where $r > 0$, for an action of Z_p is equivalent to the vector

$$(1, \overset{2h}{\ldots}, 1; a_1', a_1', \ldots, a_t', a_t', b_1, \ldots, b_u), \tag{7}$$

which is obtained by pairing the same elements of τ. Clearly $2t + u = r$ and the elements b_i are pairwise different. With these notations, we have the following lemma, proof of which is similar to the proof of Theorem 5 in [32].

Lemma 1.14. *Let τ be a generating vector for an action of Z_p of the form (7), such that $u \leq 2h + 2$. Then, the action of Z_p extends to an action of Z_{2p}.*

Proof. Since $r > 1$, we can assume using Theorem 1.3 that the generating vector for the action of $Z_p = \langle a \rangle$ is equivalent to

$$(1, \overset{2h}{\ldots}, 1; a^{n_1}, a^{n_1}, \ldots, a^{n_t}, a^{n_t}, a^{k_1}, \ldots, a^{k_u}),$$

where numbers n_i, k_i are even and $2n_1 + \ldots + 2n_t + k_1 + \ldots + k_u$ is an even multiple of p. Consider a Fuchsian group Λ, with the signature $(0; 2, \overset{s}{\ldots}, 2, p, \overset{t}{\ldots}, p, 2p, \overset{u}{\ldots}, 2p)$, where $s = 2h + 2 - u \geq 0$. Consider a tuple

$$(b^p, \overset{s}{\ldots}, b^p, (b^2)^{n_1}, \overset{t}{\ldots}, (b^2)^{n_t}, b^{k_1+p}, b^{k_2+p}, \ldots, b^{k_u+p}),$$

of elements of $Z_{2p} = \langle b \rangle$, since the sum $2n_1 + \ldots + 2n_t + k_1 + \ldots + k_u$ is an even multiple of p and $u + s$ is even, the above is a well-defined generating vector defining an action of Z_{2p}, which extends the action of Z_p by Lemma 1.10. □

Corollary 1.15. *Any action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$ where $2 \leq r \leq 2h + 2$ extends to an action of Z_{2p}.* □

The following lemma is a slightly stronger version of Theorem 3 in [32].

Lemma 1.16. *Suppose that Z_p acts on a compact Riemann surface of genus g with the signature $(h; p, \overset{r}{\ldots}, p)$ for $r \leq 2$. Then, there is an extension of this action to an action of a group of order $2p$ such that the induced action of Z_2 has the minimal number of fixed points possible.*

Proof. First, observe that there are no actions of Z_p with one fixed point. Thus, we need to consider only cases $r = 0, 2$. Observe that in both of these cases, the parity of h is the same as the parity of g, by the Riemann-Hurwitz formula.

Let $r = 0$, by Theorem 1.3, there is only one fixed point free action of Z_p. First, suppose that h is odd. and consider an action of $Z_{2p} = \langle b \rangle$ given by a $(k; -)$-generating vector

$$(b, \overset{2k}{\ldots}, b; -),$$

where $k = (h+1)/2$. By Lemma 1.4, the above action extends the action of Z_p. Moreover, the induced action of Z_2 is fixed point free, which concludes the proof in this case.

Now suppose that h is even, and consider the following $(k; 2, 2)$-generating vector for an action of $D_p = \langle b, c | b^p, c^2, cbc^{-1}b \rangle$

$$(1, \overset{2k-2}{\ldots}, 1, c, cb; c, cb^{-2}),$$

where $k = h/2$. Again, by Lemma 1.4, the above action extends the action of Z_p. Furthermore, the induced action of Z_2 has two fixed points, which is the smallest possible in this case.

By Theorem 1.3, there is only one action of Z_p with two fixed points, up to topological equivalence. Thus, the generating vector for the action of $Z_p = \langle a \rangle$ is equivalent to

$$(1, \overset{2h}{\ldots}, 1; a, a^{-1}).$$

First, suppose that h is even. Consider the following $(k; 2p, 2p)$-generating vector for an action of $Z_{2p} = \langle b \rangle$

$$(1, \overset{2k}{\ldots}, 1; b, b^{-1}),$$

where $k = (h+2)/2$. By Lemma 1.10, the above action extends the action of Z_p. Moreover, by Lemma 1.4, the induced action of Z_2 has two fixed points, which is the smallest number of fixed points of an action of Z_2 on a compact Riemann surface of even genus.

Now, assume that h is odd, and consider the following $(k;p)$-generating vector for an action of D_p

$$(1, \overset{2k-2}{\ldots}, 1, bc, c; b^{-2}),$$

where $k = (h+1)/2$. The above action extends the action of Z_p. Moreover, the induced action of Z_2 is fixed-point free. This concludes the proof. □

Lemma 1.17. *Suppose that an action of Z_p extends to an action of a group of order $2p$. Then, there exists an action of a group of order $2p$ such that the induced action of Z_2 has at most $4p-1$ fixed points.*

Proof. Observe that if Z_p is fixed-points free, the assertion holds. So, suppose that Z_p acts with the signature $(h; p, \overset{r}{\ldots}, p)$, where $r > 0$. Let G be the group of order $2p$ that extends the action of Z_p and let $(k; 2, \overset{m}{\ldots}, 2, p, \overset{t}{\ldots}, p, 2p, \overset{u}{\ldots}, 2p)$, be the signature of its action.

First, assume that G is not cyclic. Then, $u = 0$ and the generating vector for the action of Z_p is equivalent to

$$(1, \overset{2h}{\ldots}, 1; a^{n_1}, a^{-n_1}, \ldots, a^{n_t}, a^{-n_t}),$$

by Lemma 1.11. Moreover, by the Riemann-Hurwitz formula $4k + m = 2h + 2$. Let $0 \leq m' \leq 2$ be an integer such that $m' \equiv 2h + 2 \pmod 4$, and let k' be an integer such that $4k' + m' = 2h + 2$. Consider a Fuchsian group Λ with the signature

$$(k'; 2, \overset{m'}{\ldots}, 2, p, \overset{t}{\ldots}, p),$$

and the following generating vectors for actions of D_p

$$(c, b^d c, 1, \ldots, 1; b^{n_1}, \ldots, b^{n_t}),$$

if $m' = 0$, where d is an integer such that $2d = n_1 + \ldots + n_t \pmod p$, and

$$(1, \ldots, 1; c, b^n c, b^{n_1}, \ldots, b^{n_t}),$$

where $n = n_1 + \ldots + n_t$, in the remaining case.

The action of D_p defined above extends the action of Z_p. Furthermore, the number of fixed points of the induced action of Z_2 equals 0 or 2.

Now, suppose that G is cyclic. By Lemma 1.10, the generating vector of the action of Z_p is equivalent to

$$(1, \overset{2h}{\ldots}, 1; a^{n_1}, a^{n_1}, \ldots, a^{n_t}, a^{n_t}, a^{k_1}, \ldots, a^{k_u}).$$

By Theorem 1.3, we may assume that the numbers n_i, k_i are even. By the Riemann-Hurwitz formula $2h+2 = 4k+m+u$. Clearly, we may assume that $u \leq p-1$, since every

generating vector can be brought to the form (7). Let $0 \leq m' \leq 3$ and k' be integers such that $2h+2 = 4k' + m' + u$, observe that m' is even if and only if u is even. Consider a Fuchsian group with the signature $(k'; 2, \overset{m'}{\ldots}, 2, p, \overset{t}{\ldots}, p, 2p, \overset{u}{\ldots}, 2p)$ and the following generating vector for an action of Z_{2p}

$$(1, \overset{2k'}{\ldots}, 1; b^p, \overset{m'}{\ldots}, b^p, (b^2)^{n_1}, \overset{t}{\ldots}, (b^2)^{n_t}, b^{k_1+p}, \overset{u}{\ldots}, b^{k_u+p}).$$

First, observe that the above is a well-defined generating vector. Firstly, the orders of generators are preserved since n_i's and k_i's are even. Secondly, notice that for

$$z = m'p + 2n_1 + \ldots + 2n_t + k_1 + \ldots + k_u + up,$$

we have $b^z = 1$, since $2(n_1 + \ldots + n_t) + k_1 + \ldots + k_u$ is an even multiple of p and $m' + u$ is even. The action of Z_{2p} defined above extends the action of Z_p by Lemma 1.10. Finally, by Lemma 1.4, the number of fixed points of the induced action of Z_2 equals at most $4p-1$ since $u \leq p-1$. This concludes the proof. □

1.5. **The moduli space of compact Riemann surfaces and its singular locus.** Consider an abstract group Λ with the presentation (1), and a set of monomorphisms

$$\mathcal{W}(\Lambda) = \{\rho\colon \Lambda \hookrightarrow \mathrm{Isom}(\mathcal{H}) = \mathrm{PSL}(2,\mathbb{R}) \ : \ \rho(\Lambda) \ \text{ is a Fuchsian group}\}$$

called the Weil space of Λ. This set admits a natural compact-open topology since Λ is a discrete group and $\mathrm{PSL}(2,\mathbb{R})$ has the structure of a Lie group. We define an equivalence relation $\sim$ on $\mathcal{W}(\Lambda)$ as follows $\rho_1 \sim \rho_2$ if and only if there exists $\lambda \in \mathrm{Isom}(\mathcal{H})$ such that $\rho_1(x) = \lambda\rho_2(x)\lambda^{-1}$ for every $x \in \Lambda$. The Teichmüller space $\mathcal{T}(\Lambda)$ is the quotient space $\mathcal{W}(\Lambda)/\sim$. Fricke and Klein proved in [17], that the Teichmüller space, together with the topology obtained from the Weil space, is homeomorphic to $\mathbb{C}^{d(\Lambda)}$, where $d(\Lambda) = 3(h-1)+r$. We will call the number $d(\Lambda)$ the dimension of Λ. There is a natural action of the mapping class group $\mathfrak{M}(\Lambda) = \mathrm{Out}(\Lambda)$ on the Teichmüller space defined as

$$([\varphi], [\rho]) \mapsto [\rho \circ \varphi^{-1}].$$

The orbit space $\mathcal{M}(\Lambda)$ of this action is called the moduli space of Λ. The moduli space of compact Riemann surfaces of genus g is obtained by following the above construction for a group Λ isomorphic to the fundamental group of closed oriented surface of genus g. We shall call the topology of $\mathcal{M}_g$ inherited from $\mathcal{T}_g$, the Teichmüller topology of the moduli space. As mentioned in the introduction there are small discrepancies. Still, we can now define the singular locus $\mathcal{S}_g$ of $\mathcal{M}_g$ as the subset composed of points that represent the compact Riemann surfaces with nontrivial automorphisms.

In our work, we shall use a more technical, but better suited for our purposes, definition of the Broughton stratification than the one used in [11]. So, let $\theta\colon \Lambda \to G$ be a smooth epimorphism defining an action of a group G on a compact Riemann surface of genus g. The *equisymmetric stratum*, or shortly *stratum*, $\mathcal{M}_g^{G,\theta}$ is a set of points of $\mathcal{M}_g$ representing isomorphism classes of compact Riemann surfaces with the action of the

full automorphism group topologically equivalent to the action of G defined by θ. The Zariski closure $\overline{\mathcal{M}}_g^{G,\theta}$ of a nonempty stratum is composed of points of $\mathcal{M}_g$ representing the classes of compact Riemann surfaces for which the action of its full automorphism group extends the action of G. We present the basic properties of equisymmetric stratification discovered by Broughton in [11]

- Strata $\mathcal{M}_g^{G,\theta}$ are smooth, locally closed, connected algebraic subvarieties of the moduli space,
- Zariski closures of nonempty strata are irreducible algebraic subvarieties of the moduli space.

The Zariski closures of the equisymmetric strata are also closed in the topology of $\mathcal{M}_g$ inherited from the Teichmüller space.

We want to emphasize here a strong relation between the Cornalba components defined in [13] and Zariski closures of equisymmetric strata. That is, every Cornalba component is a Zariski closure of a stratum $\mathcal{M}_g^{Z_p,\theta}$. We underline here that Cornalba components are irreducible subvarieties of the moduli space.

A smooth epimorphism $\theta\colon \Lambda \to G$ defines an embedding $i\colon \Gamma \to \Lambda$ of an abstract group isomorphic to $\ker\theta$ into Λ viewed as an abstract group. By [18], i induces an embedding

$$j\colon \mathcal{T}(\Lambda) \to \mathcal{T}(\Gamma) = \mathcal{T}_g;\ \ [\rho] \mapsto [\rho \circ i].$$

The image of $j(\mathcal{T}(\Lambda))$ under the canonical projection $\pi\colon \mathcal{T}_g \to \mathcal{M}_g$ is equal to the Zariski closure of the stratum $\mathcal{M}_g^{G,\theta}$. Since every Teichmüller space is connected we get the following

Proposition 1.18. *Zariski closure $\overline{\mathcal{M}}_g^{G,\theta}$ of a nonempty stratum is connected in the Teichmüller topology of $\mathcal{M}_g$.*

We say that two strata $\mathcal{M}_g^{G,\theta}$ and $\mathcal{M}_g^{H,\rho}$ are separated if and only if

$$\overline{\mathcal{M}}_g^{G,\theta} \cap \mathcal{M}_g^{H,\rho} = \mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{H,\rho} = \emptyset.$$

Furthermore, we say that they are connected by a sequence of strata if and only if there exists a sequence of strata $\mathcal{M}_g^{G_i,\theta_i}$ for $i = 1, \ldots, n$ such that:

$$(8)\qquad \begin{cases} \bullet\ \ \mathcal{M}_g^{G,\theta} = \mathcal{M}_g^{G_1,\theta_1}, \\ \bullet\ \ \mathcal{M}_g^{H,\rho} = \mathcal{M}_g^{G_n,\theta_n}, \\ \bullet\ \ \text{strata } \mathcal{M}_g^{G_i,\theta_i} \text{ and } \mathcal{M}_g^{G_{i+1},\theta_{i+1}} \text{ are not separated for arbitrary } i. \end{cases}$$

A stratum that is separated from every other stratum is called *isolated*. These strata were studied extensively in several papers cf. [28], [14]. Below we present a theorem that characterizes their structure. We proceed the theorem with a technical lemma.

Lemma 1.19. *A stratum defined by an action of a group of order smaller than 4 on a compact Riemann surface of genus greater than 5 is never empty.*

Proof. Assume, to get a contradiction, that an action of a group G of order 2 or 3 defines an empty stratum. Then, the signature of this action corresponds to a non-finitely maximal Fuchsian group. Thus, its signature has to occur on Singerman's list of nonmaximal Fuchsian groups [36]. So, it is equal to $(2;-)$, $(1;p,p)$ or $(0;p,p,p,p)$, which is impossible by the Riemann-Hurwitz formula. □

Theorem 1.20. *Let $g \geq 5$ be an integer. A nonempty stratum $\mathcal{M}_g^{G,\theta}$ is isolated if and only if it is Zariski-closed and G has prime order.*

Proof. The sufficiency is obvious. For the necessity suppose that a nonempty stratum $\mathcal{M}_g^{G,\theta}$ is isolated. Clearly, it has to be Zariski-closed. For the second part, let H be a subgroup of G of prime order. If the induced action of H defines a nonempty stratum, then

$$\mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{H,\rho} \neq \emptyset,$$

and thus $\mathcal{M}_g^{G,\theta} = \mathcal{M}_g^{H,\rho}$ since $\mathcal{M}_g^{G,\theta}$ is isolated. So, suppose that the stratum defined by the action of H is empty. In this case, the action of H is extended by an action of a group L of order divisible by 2 or 3. It is important to note that in this case, if a compact Riemann surface admits the action of H, then this surface also admits the action of L. Let S be a compact Riemann surface such that $[S] \in \mathcal{M}_g^{G,\theta}$, i.e. the action of its full automorphism group is equivalent to the action of G. By the above observation, the order of $\mathrm{Aut}(S) \cong G$ is divisible by 2 or 3. Let K be a subgroup of G having the order equal to 2 or 3. The stratum defined by the action of K is nonempty by Lemma 1.19. Then

$$\mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{K,\tau} \neq \emptyset,$$

which is a contradiction. Thus, the action of H defines a nonempty stratum, which concludes the proof. □

Corollary 1.21. *Suppose that $\tau\colon \Lambda \to Z_p$ is a smooth epimorphism defining a finitely maximal action of Z_p on a compact Riemann surface of genus g. Then, the corresponding stratum $\mathcal{M}_g^{Z_p,\tau}$ is isolated.* □

The following lemma may seem trivial, but it would be useful to formulate here this immediate consequence of the definition of the equisymmetric stratification.

Lemma 1.22. *If $\mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{H,\rho} \neq \emptyset$, then H is a subgroup of G.*

Proof. Let S be a Riemann surface defining a point $[S]$ in $\mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{H,\rho}$. Then, by the definition of the equisymmetric stratum, G is the full group of automorphism of S since $[S] \in \mathcal{M}_g^{G,\theta}$. But $[S] \in \overline{\mathcal{M}}_g^{H,\rho}$, and so H is an automorphisms group of S by the definition of the Zariski closure, which ends the proof. □

Proposition 1.23. *Let $\mathcal{C}$ be a connected component of the singular locus in the Teichmüller or the Zariski topology. A stratum $\mathcal{M}_g^{G,\theta}$ is contained in $\mathcal{C}$ if and only if it is connected by a sequence of strata with some stratum $\mathcal{M}_g^{H,\rho}$ contained in $\mathcal{C}$.*

Proof. We give the proof for the Teichmüller topology; the proof for the Zariski topology is analogous.

Assume first, that strata $\mathcal{M}_g^{G,\theta}$ and $\mathcal{M}_g^{H,\rho}$ are connected by a sequence of strata $\mathcal{M}_g^{G_i,\theta_i}$. Then the set

$$\bigcup_{i=1}^{n} \overline{\mathcal{M}}_g^{G_i,\theta_i}$$

is connected by Proposition 1.18, which means that it is contained in $\mathcal{C}$. Therefore, $\mathcal{M}_g^{G,\theta}$ and $\mathcal{M}_g^{H,\rho}$ are contained in $\mathcal{C}$.

Assume that strata $\mathcal{M}_g^{G,\theta}$ and $\mathcal{M}_g^{H,\rho}$ are contained in $\mathcal{C}$ and are not connected by a sequence of strata. Recall that there are only finitely many strata, and by the results of [11], the Zariski closure of a stratum is closed in the Teichmüller topology. Let A be the set of Zariski closures of all strata that are connected by a sequence of strata with $\mathcal{M}_g^{G,\theta}$ and let B be the set of Zariski closures of all other strata contained in $\mathcal{C}$. Observe that the sets $\mathcal{A}$ and $\mathcal{B}$, which are the sums of elements of A and B respectively are nonempty and closed, as finite sums of closed sets. Moreover, these sets are disjoint, since otherwise there would exist elements $\overline{\mathcal{M}}_g^{K,\alpha}$ of A, and $\overline{\mathcal{M}}_g^{L,\beta}$ of B with a nonempty intersection. But this would mean that $\mathcal{M}_g^{L,\beta}$ is connected by a sequence of strata with $\mathcal{M}_g^{G,\theta}$ which is impossible. Finally, observe that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ which is a contradiction. This concludes the proof. □

Corollary 1.24. *Suppose that a stratum $\mathcal{M}_g^{Z_p,\tau}$ is isolated. Then, it is a connected component of the singular locus.* □

The following main Theorem of this section is a prompt consequence of the above results.

Theorem 1.25. *The set of connected components of $\mathcal{S}_g$ in the Teichmüller and the Zariski topology coincides.* □

To end this subsection, we summarize the results of Kulkarni [28], which will be useful for our purposes.

Lemma 1.26. *Let $g > 3$ be an integer. Then, $\mathcal{S}_g$ contains an isolated point if and only if $2g + 1$ is prime.* □

1.6. **Combinatorial density of a subset with respect to a filtration.** Throughout the subsection X will denote a countable set. By a filtration $\mathcal{X}$ of X we understand a ascending family of finite subsets of X

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

covering X.

Let Y be a subset of X. We will say that Y is *combinatorially dense in* X with respect to filtration $\mathcal{X}$ if and only if

$$\lim_{n\to\infty} \frac{|X_n \cap Y|}{|X_n|} = 1. \tag{9}$$

The above limit, if it exists, can be viewed as a proportion between the set X and its subset Y with respect to a filtration of X. We present this by the following example.

Example 1.27. Let $X = \mathbb{Z}$, $Y = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ and consider the following three filtrations of X:

$$X_n^1 = \{-n, \ldots, -2, -1, 0, 1, 2, \ldots, n^2\}, \quad X_n^2 = \{-n, \ldots, -2, -1, 0, 1, 2 \ldots, n\} \text{ and}$$

$$X_n^3 = \{-n, \ldots, -2, -1, 0, 1, 2 \ldots, k\},$$

where $2^{k-1} < n \leq 2^k$. Then

$$\lim_{n\to\infty} \frac{|X_n^i \cap Y|}{|X_n^i|} = 1, 1/2, \text{ and } 0$$

for $i = 1, 2, 3$ respectively.

2. Structure of ordinary connected components of the singular locus

The main purpose of this section is to describe the structure of ordinary connected components of the singular locus in terms of equisymmetric stratification. The results of this paper were published in [37]. We start with a technical lemma.

Lemma 2.1. *Let $\theta\colon \Lambda \to G$ be a smooth epimorphism defining an action of G on a compact Riemann surface of genus $g \geq 5$. Suppose that the induced action of a non-trivial subgroup H of G defines an empty stratum. Then, $\mathcal{M}_g^{G,\theta}$ is a subset of the exceptional component.*

Proof. First, notice that if the stratum defined by the action of G is empty, then there is nothing to do. So suppose that the stratum $\mathcal{M}_g^{G,\theta}$ is not empty, and observe that, by Lemma 1.19, it is sufficient to show that the order of G is divisible by 2 or 3, since then there exists a subgroup K of G of order 2 or 3 defining a nonempty stratum, and

$$\mathcal{M}_g^{G,\theta} \cap \overline{\mathcal{M}}_g^{K,\tau} \neq \emptyset.$$

So, let Δ be a subgroup of Λ such that $\Delta/\ker\theta = H$. Since the action of H defines an empty stratum, Δ is contained in a Fuchsian group Δ' containing $\ker\theta$ as a normal subgroup such that the dimensions of the corresponding Teichmüller spaces are equal (cf. [12], where similar reasoning was used). By [36] the index of Δ in Δ' is divisible by 2 or 3. Moreover, Δ' is a subgroup of Λ. Otherwise, the action of G would define an empty stratum, which is not our case. Therefore, the order of G is divisible by 2 or 3. □

Observe that every connected component $\mathcal{C}$ of the singular locus contains a stratum defined by an action of a group of prime order. If $\mathcal{C}$ is an ordinary component, then the stratum $\mathcal{M}_g^{Z_p,\tau}$ contained in $\mathcal{C}$ satisfies the assumptions of Lemma 1.6, since otherwise it would be contained in the exceptional component, by Lemma 1.7. Moreover, if $\mathcal{C}$ consists of more than one stratum, then $\mathcal{M}_g^{Z_p,\tau}$ is not Zariski-closed, since otherwise it would be isolated. Therefore, the action of Z_p extends to an action of a group of order pq for some prime number q, containing it as a normal subgroup. In the following lemmas, we deal with the two out of four possible cases.

Lemma 2.2. *Assume that an action of Z_p on a compact Riemann surface of genus $g \geq 5$ extends to an action of a non-abelian semidirect product $Z_p \rtimes Z_q$, for a prime number $q \neq p$. Then, the corresponding stratum $\mathcal{M}_g^{Z_p,\tau}$ is contained in the exceptional component.*

Proof. Let H be a subgroup of $Z_p \rtimes Z_q$ of order q. By Lemma 2.1, we may assume that the stratum defined by the induced action of H is not empty. Observe that H is not normal in $Z_p \rtimes Z_q$, and thus it coincides with its normalizer. This concludes the proof by Corollary 1.8. □

Lemma 2.3. *Assume that an action of Z_p on a compact Riemann surface of genus $g \geq 5$ extends to an action of Z_{pq} for some prime number $q \neq p$. Then, the corresponding equisymmetric stratum $\mathcal{M}_g^{Z_p,\tau}$ is contained in the exceptional component.*

Proof. Let $Z_{pq} = \langle a \rangle$ act with the signature

$$(k; p, \overset{s}{\dots}, p, q, \overset{t}{\dots}, q, pq, \overset{u}{\dots}, pq).$$

By Lemma 2.1, we may assume that the strata defined by induced actions of subgroups of orders p and q are nonempty. Moreover, by Lemma 1.7, we may suppose that the orbit genera of the actions of these subgroups satisfy $h_1 < (p-3)/2$ and $h_2 < (q-3)/2$ respectively. Without loss of generality, we may assume that $p < q$. By the Riemann-Hurwitz formula

$$h_1 = kq + (t + u - 2)(q-1)/2. \tag{10}$$

Notice that if $t + u = 0$, then the action of a subgroup of order q is fixed point free by Lemma 1.4, which would conclude the proof by Lemma 1.7. So, suppose that $t + u \neq 0$, and observe that then $t + u \leq 2$ since otherwise $h_1 \geq (q-1)/2$ by (10), which contradicts our assumption on h_1. Furthermore, observe that $t + u \neq 1$ since the product of elements of order p in Z_{pq} cannot have order q or pq. Therefore, $h_1 = k = 0$, since otherwise $h_1 \geq q$, which is not our case. Thus, the subgroup of order q acts with 2, $p+1$ or $2p$ fixed points if $u = 2, 1, 0$ respectively. We may assume that $u \neq 2$, by Lemma 1.7. In the remaining cases, the generating vector for the action of the subgroup of order q generated by a^p is equivalent to

$$(1, \overset{2n}{\dots}, 1, a^p, \overset{p}{\dots}, a^p, a^{-p^2}) \quad \text{or} \quad (1, \overset{2n}{\dots}, 1, a^p, \overset{p}{\dots}, a^p, a^{-p}, \overset{p}{\dots}, a^{-p}),$$

by Lemma 1.10. We will show that these actions extend to actions of the group $Z_{2q} = \langle b \rangle$. Consider a Fuchsian group with the signature

$$(0; 2, \overset{2n}{\dots}, 2, q, \overset{v}{\dots}, q, 2q, 2q),$$

where $v = (p-1)/2$, and a generating vector

$$(b^q, \overset{2n}{\dots}, b^q, b^2, \overset{v}{\dots}, b^2, b, b^{-p}).$$

The action defined above extends the action of Z_q by Lemma 1.10 and Theorem 1.3. For the second case, consider a Fuchsian group with the signature

$$(0; 2, \overset{2n}{\dots}, 2, q, \overset{v}{\dots}, q, q, \overset{v}{\dots}, q, 2q, 2q)$$

and a generating vector

$$(b^q, \overset{2n}{\dots}, b^q, b^2, \overset{v}{\dots}, b^2, b^{-2}, \overset{v}{\dots}, b^{-2}, b, b^{-1}),$$

which is clearly the extension we are looking for. □

The following technical lemma will be used in the proof of the upcoming theorem. The lemma is well-known, and its proof can be found in [22].

Lemma 2.4. *Let Λ_1 and Λ_2 be Fuchsian groups with orbit genera h_1 and h_2 respectively, such that $\Lambda_1 \trianglelefteq \Lambda_2$. Then $h_1 \geq h_2$.* □

Theorem 2.5. *Let G be a p-group of order p^n for $n \geq 2$, and let $\theta\colon \Lambda \to G$ be a smooth epimorphism defining a stratum $\mathcal{M}_g^{G,\theta}$, not contained in the exceptional component. Then, G is cyclic and acts with the signature $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$, for some integer s.*

Proof. We will prove the theorem by induction on n. So, suppose that $n = 2$. We will show first that G is cyclic. Assume, to get a contradiction, that $G = Z_p^2$. Let H be a non-trivial cyclic subgroup of G, acting with the orbit genus $h < (p-3)/2$ by Lemma 1.7. Let

$$(k; p, \overset{u}{\ldots}, p, p, \overset{t}{\ldots}, p),$$

be the signature of the action of G, where the last t periods correspond to these elliptic generators of Λ, whose images, under θ, do not belong to H. Notice that $t \neq 0$, since otherwise, every cyclic subgroup different from H would act without fixed points, which is not our case by Lemma 1.7. By the Riemann-Hurwitz formula and Lemma 1.4

$$h = kp + (t-2)(p-1)/2. \tag{11}$$

Since $t \neq 0$ and $h < (p-3)/2$, the only possibility is that $t = 2$ and $h = k = 0$. Hence, the signature of the action of G equals $(0; p, \overset{u}{\ldots}, p, p, p)$. Which means that there are cyclic subgroups of G acting without fixed points, since G has $p+1$ cyclic subgroups. This is impossible by Lemma 1.7 again. Therefore, G is cyclic.

Now, suppose that $G = Z_{p^2} = \langle a \rangle$ and the signature of its action equals $(k; p, \overset{u}{\ldots}, p, p^2, \overset{t}{\ldots}, p^2)$. Let H be the subgroup of G of order p. By the Riemann-Hurwitz formula and Lemma 1.4 again, the orbit genus h of the action of H satisfies (11). Moreover, $h < (p-3)/2$ by Lemma 1.7 once more. We start by ruling out the case $t = 0$. So, assume that $t = 0$, and observe that in this situation $k = h = 1$, since k cannot equal 0, and if $k \geq 2$, then $h > p$, which is not our case. Thus, the signature of the action of G is $(1; p, \overset{u}{\ldots}, p)$. Let

$$(a_1, a_2; a^{n_1 p}, \ldots, a^{n_u p}),$$

for some $a_1, a_2 \in G$, be the generating vector for the action of G. It is easy to see that the number $n_1 + \ldots + n_u$ is divisible by p. Let $b = a^p$, by Lemma 1.10, the generating vector corresponding to the action of H is equivalent to

$$(1, 1; b^{n_1}, \overset{p}{\ldots}, b^{n_1}, \ldots, b^{n_u}, \overset{p}{\ldots}, b^{n_u}). \tag{12}$$

Consider, an action of a group $G' = Z_p^2 = \langle c, d | c^p, d^p, [c,d] \rangle$ on a compact Riemann surface of genus g defined by the following $(1; p, \overset{u}{\ldots}, p)$-generating vector

$$(d, d; c^{n_1}, \ldots, c^{n_u}).$$

By Lemma 1.4, the subgroup $L = \langle d \rangle$ acts without fixed points, so it extends to an action of a group L' of even order by Lemma 1.7. Furthermore, the generating vector for the

action of $\langle c\rangle$ is equivalent to (12), by Lemma 1.12 and Theorem 1.3. Let K be a subgroup of L' of order 2. Then the following sequence of strata

$$\mathcal{M}_g^{G,\theta},\ \mathcal{M}_g^{H},\ \mathcal{M}_g^{G'},\ \mathcal{M}_g^{L},\ \mathcal{M}_g^{L'},\ \mathcal{M}_g^{K},$$

satisfies the properties of (8), hence the stratum $\mathcal{M}_g^{G,\theta}$ is contained in the exceptional component, which is a contradiction. Thus $t \neq 0$. Using the same arguments as above, we can see that the only possibility is that $t = 2$ and $k = h = 0$. This concludes the proof in the case $n = 2$.

Now, suppose that $n > 2$ and let H be a normal subgroup of G of order p^{n-1}. By Lemma 2.1, the action of H defines a nonempty stratum, which is not contained in the exceptional component. So, by the induction H is a cyclic group acting with the signature $(0; p, \overset{s'}{\ldots}, p, p^{n-1}, p^{n-1})$, for some integer s'. Thus, the orbit genus of the action of G is also 0, by Lemma 2.4. Suppose that G is not cyclic. In particular, this means that every period in the signature of the action of G is smaller than p^n. Let b be a generator of H. By Lemma 1.4 b has two fixed points. However, using Lemma 1.4 once more for b viewed as an element of G we can see that the number of its fixed points is divisible by p, since every period of the signature of G is smaller than p^n and H is normal in G. This contradiction proves that G is cyclic. Finally, observe that every automorphism from H of order greater than p has precisely two fixed points, so the signature of the action of G has to be equal to $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$ by Lemma 1.4. This concludes the proof. □

We are in a position to state and prove the main result of this part.

Theorem 2.6. *Let $\mathcal{C}$ be an ordinary connected component of $\mathcal{S}_g$. Then, $\mathcal{C}$ is the Zariski closure of some stratum defined by an action of Z_p. Moreover, for every stratum $\mathcal{M}_g^{G,\theta}$ contained in $\mathcal{C}$, the group G is a cyclic p-group.*

Proof. First, notice that if $\mathcal{C}$ is composed of one stratum, then the assertion follows immediately. So, suppose that $\mathcal{C}$ consists of more than one stratum, and let $\mathcal{M}_g^{G,\theta}$ be one of them. By Theorem 2.5 every Sylow subgroup of G is cyclic. Moreover, by the Feit-Thompson Theorem, G is solvable since it has odd order. Let $D_1 \triangleright D_2 \triangleright \ldots \triangleright D_n \triangleright \{1\}$ be its derived series. Clearly, D_n is an abelian group, and so it is a direct sum of cyclic p-groups. But, since every Sylow subgroup of G is cyclic, D_n is also cyclic. Let Z_p be a subgroup of D_n. Since D_n is characteristic in G, Z_p is a normal subgroup of G. We will show that G is a p-group. Assume that a prime number $q \neq p$ divides the order of $\widetilde{G} = G/Z_p$. Then, by the Sylow Theorem, $\widetilde{G}$ contains a subgroup $\widetilde{Q}$ of order q. Thus, the subgroup K of G such that $\widetilde{Q} = K/Z_p$ has an order equal to pq and contains H as a normal subgroup, which is impossible by Lemmas 2.2 and 2.3. So G is a cyclic p-group by Theorem 2.5.

Now we will prove that the above-defined action of Z_p is the one we are looking for. Let $\mathcal{M}_g^{L,\rho}$ be another stratum contained in $\mathcal{C}$. First, we will show that L is a p-group. By

the above argument, L is a cyclic q-group. Strata $\mathcal{M}_g^{Z_p,\tau}$ and $\mathcal{M}_g^{L,\rho}$ are contained in the same connected component, so they are connected by a sequence of strata $\mathcal{M}_g^{G_i,\theta_i}$ such that

$$\overline{\mathcal{M}}_g^{G_i,\theta_i} \cap \mathcal{M}_g^{G_{i+1},\theta_{i+1}} \neq \emptyset \text{ or } \mathcal{M}_g^{G_i,\theta_i} \cap \overline{\mathcal{M}}_g^{G_{i+1},\theta_{i+1}} \neq \emptyset. \tag{13}$$

Clearly, every group G_i is a cyclic p_i-group for some prime number p_i. By (13) $G_i \leq G_{i+1}$ or $G_{i+1} \leq G_i$, thus L is a p-group. Moreover, observe that since every group G_i is cyclic, the induced actions of subgroups of order p of G_i and G_{i+1} are topologically equivalent by (13), which concludes the proof. □

Recall that Cornalba components coincide with the Zariski closures of strata defined by actions of prime-ordered groups. This allows us to see the following immediately.

Corollary 2.7. *Every ordinary connected component $\mathcal{C}$ of the singular locus coincides with a Cornalba Component; in particular, $\mathcal{C}$ is an irreducible subvariety of the moduli space.* □

The following corollary is a prompt consequence of the above Theorem and Lemma 1.10.

Corollary 2.8. *Let $\mathcal{C}$ be an ordinary connected component of $\mathcal{S}_g$ consisting of more than one stratum and being the Zariski closure of a stratum $\mathcal{M}_g^{Z_p,\tau}$. Then Z_p acts with the signature $(0; p, \overset{r}{\ldots}, p)$, where $r \equiv 2 \pmod p$ and the generating vector for its action is equivalent to*

$$(a, \overset{p}{\ldots}, a, a^{n_2}, \overset{p}{\ldots}, a^{n_2}, \ldots, a^{n_s}, \overset{p}{\ldots}, a^{n_s}, a^k, a^{-k}). \tag{14}$$

where $r = sp + 2$ and $Z_p = \langle a \rangle$. □

Remarks 2.9. The above conditions for a stratum $\mathcal{M}_g^{Z_p,\tau}$ to be contained in an ordinary component are only necessary. Consider, for example, an action of $Z_p = \langle a \rangle$ defined by a $(0; p, \overset{2p+2}{\ldots}, p)$-generating vector:

$$(a, \overset{p}{\ldots}, a, a^{-1}, \overset{p}{\ldots}, a^{-1}, a^2, a^{-2}),$$

and an action of a group $D_p = \langle a, b | a^p, b^2, bab^{-1}a \rangle$ given by a $(0; 2, 2, p, \overset{p+1}{\ldots}, p)$-generating vector:

$$(a^{-2}b, b, a, \overset{p}{\ldots}, a, a^2).$$

By Lemma 1.11 and Theorem 1.3, the latter action extends the former one, and therefore the corresponding stratum $\mathcal{M}_g^{Z_p,\tau}$ is contained in the exceptional component.

The following Theorem gives necessary and sufficient conditions for a strutum defined by an action given by a generating vector of the form (14) to be contained in an ordinary connected component. It is a stronger version of Theorem 3.11 from [37], the proof presented below was found after the publication of [37].

Theorem 2.10. *Let τ be a generating vector of the form* (14) *defining an action of a cyclic group $Z_p = \langle a \rangle$ on a compact Riemann surface of genus $g \geq 5$. Then, the corresponding equisymmetric stratum $\mathcal{M}_g^{Z_p,\tau}$ is contained in $\mathcal{C}_g^{2,3}$ if and only if, the action of Z_p extends to an action of a cyclic group Z_{pq} for some prime $q \neq p$ or to an action of a dihedral group D_p.*

Proof. The sufficiency follows from Lemmas 2.2 and 2.3. So, suppose that the stratum $\mathcal{M}_g^{Z_p,\tau}$ is contained in the exceptional component. Let $\mathcal{M}_g^{G_i,\theta_i}$ be a sequence of strata connecting the stratum $\mathcal{M}_g^{Z_p,\tau}$ with a stratum $\mathcal{M}_g^{Z_2,\psi}$. As in the proof of Theorem 2.6

$$G_{i-1} < G_i \text{ or } G_i < G_{i-1}$$

which we will extensively use throughout the proof.

First, we will show that the action of Z_p extends to an action of a group of order pq for some $q \neq p$. Let k be the minimal subscript for which the group G_k has a trivial Sylow p-subgroup. Such a number exists since $G_n = Z_2$. Moreover, $k > 2$ since $G_1 = Z_p$ and $G_1 < G_2$. Assume that there is $i < k$ such that G_i does not extend the action of Z_p. Notice that $i > 1$. If G_{i-1} is a subgroup of G_i, then G_i extends the action defined by τ, by the definition of i. So, $G_i \leq G_{i-1}$. Since $i < k$, the action of G_i has a nontrivial Sylow p-subgroup. So, the Sylow p-subgroup of G_{i-1} contains two distinct subgroups of order p, since G_{i-1} extends the action of Z_p and G_i. Let p^t be an order of the sylow p-subgroup S of G_{i-1}. By the above observation, S is not cyclic, hence every period in the signature of its action is a power of p smaller than p^t. Let $P = \langle \varphi \rangle$ be a subgroup of S whose action is equivalent to the action of Z_p defined by τ. By Lemma 1.4, the number of fixed points of φ is divisible by p. To see this, suppose that $p^{t'}$ is a period in the signature of S corresponding to an elliptic generator, whose power of image is conjugate to φ in S. Then, P is contained in a cyclic group C of order $p^{t'}$. But every proper subgroup of S is subnormal, so the normalizer of this subgroup in S has order greater than $p^{t'}$. Since every subgroup of a cyclic group is characteristic, the normalizer of P in S contains the normalizer C. So, every element of the sum from Lemma 1.4 is a non-trivial power of p. This is a contradiction. Thus, G_i extends the action of Z_p for every $i < k$, in particular G_{k-1} extends this action.

Notice that $G_k < G_{k-1}$, since the order of G_k is coprime with p. First, suppose that $s \geq 2$. By the Castelnuovo-Severi inequality (3) G_{k-1} contains the unique p-gonal subgroup H, whose action is clearly equivalent to the action defined by τ. Hence, there is a prime number $q \neq p$ dividing the order of G_{k-1}/H, since G_{k-1} extends the action of G_k. So, H is a normal subgroup of a group of order pq. Now, suppose that $s = 1$. Then, (14) is equivalent to

$$(a, \overset{p}{\ldots}, a, a^k, a^{-k}).$$

If p^2 does not divide the order of G_{k-1}, then the action of Z_p extends to an action of a group of order pq by Lemma 1.6. So, suppose that p^2 divides the order of G_{k-1}. By

Sylow Theorems there exists a subgroup L of G_{k-1} of order p^2 extending the action of Z_p. Clearly, L is cyclic. Moreover, observe that L acts with the signature $(0; p, p^2, p^2)$ by Lemma 1.4 and Lemma 1.9. Since G_{k-1} extends the action of G_k, $L \neq G_{k-1}$. This means that the signature of the action of G_{k-1} has Teichmüller dimension 0. Hence, it appears on the Singerman's list from [36]. But the only possibility is that the signature of G_{k-1} equals $(0; 2, 2p, 2p^2)$ and L is a subgroup of G_{k-1} of index 2. Hence, it is normal in G_{k-1}. Thus, the subgroup of order p of L is also normal in G_{k-1}. Therefore, the action of Z_p extends to an action of a group of order $2p$. This concludes the first part of the proof.

Let K be a group of order pq, for $q \neq p$, extending the action of Z_p. Moreover, assume that $q > 2$ and K is not cyclic. Thus

$$K = Z_p \rtimes Z_q = \langle a, b \,|\, a^p, b^q, bab^{-1}a^{-\lambda} \rangle$$

where λ is a qth-root of the unity modulo p and τ is equivalent to

$$(a^{k_1}, a^{k_1\lambda}, \ldots, a^{k_1\lambda^{q-1}}, \ldots, a^{k_l}, a^{k_l\lambda}, \ldots a^{k_l\lambda^{q-1}}) \tag{15}$$

by Lemma 1.11 and Theorem 1.3. Recall however that by our assumption τ is equivalent to

$$(a^{n_1}, \overset{p}{\ldots}, a^{n_1}, a^{a_2}, \overset{p}{\ldots}, a^{n_2}, \ldots, a^{n_s}, \overset{p}{\ldots}, a^{n_s}, a^k, a^{-k}).$$

So, without loss of generality, we can assume that, up to a permutation, these two vectors coincide. Moreover, we may assume that $n_1 = k_1 = 1$. The vector (15) can be decomposed into orbits of the action of Z_q on $a, a^{k_2}, \ldots, a^{k_l} \in Z_p$. First, suppose that $a^k = a^{n_i}$, for some i. Then the element a^k occurs $zp + 1$ times in τ since $k \neq -k \pmod p$. So, there are $zp + 1$ orbits in (15) containing a^k. Furthermore, since $q > 2$, there are at least two other elements contained in these orbits. Which is a contradiction since otherwise there would be more than two elements of Z_p occurring $zp + 1$ times in τ. But every element different than a^k and a^{-k}, appears mp times in τ.

Now suppose, that $a^k \neq a^{n_i}$, for every i. Observe that since orbits of the action of Z_q on Z_p have at least three elements, there exists some j for which a^{n_j} and a^k are contained in the same orbit, which is impossible since otherwise a^k would occur in τ more than once. This concludes the proof. □

Remarks 2.11. The natural question is whether the prime number q in the above Theorem may be assumed to be equal to 2 or 3, i.e. whether the stratum $\mathcal{M}_g^{Z_p,\tau}$ is connected *directly* to a stratum defined by an action of Z_2 or Z_3. It appears that it cannot be done. To see this consider an action of $Z_p = \langle a \rangle$, for $p > 5$ defined by a $(0; p, \overset{10p+2}{\ldots}, p)$-generating vector

$$(a, \overset{5p}{\ldots}, a, a^2, \overset{5p}{\ldots}, a^2, a^3, a^{-3}),$$

and an action of $Z_{5p} = \langle b \rangle$ given by a $(0; p, \overset{2p}{\ldots}, p, 5p, 5p)$-generating vector:

$$(b^5, \overset{p}{\ldots}, b^5, b^{10}, \overset{p}{\ldots}, b^{10}, b^3, b^{-3}).$$

By Lemma 1.10, the action of Z_{5p} extends the action of Z_p, thus the corresponding stratum $\mathcal{M}_g^{Z_p}$ is contained in the exceptional component by Theorem 2.10. However, by Lemmas 1.9, 1.10, 1.11 and Theorem 1.3, the action of Z_p does not extend to an action of Z_{2p}, Z_{3p} nor D_p.

Lemma 2.12. *Let Δ be a Fuchsian group with the signature $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$. There is a unique Fuchsian subgroup Λ of Δ with the signature $(0; p, \overset{v}{\ldots}, p)$ for $v = sp^{n-1} + 2$.*

Proof. Observe that by the results of Harvey from [24], there exists a smooth epimorphism $\theta\colon \Delta \to Z_{p^n}$. By Lemma 1.4 and the Riemann-Hurwitz formula the Fuchsian group $\Lambda = \theta^{-1}(Z_p)$ has the signature $(0; p, \overset{v}{\ldots}, p)$.

Let Λ' be another Fuchsian subgroup of Δ with this signature. By the smoothness of θ and uniqueness of Z_p in Z_{p^n} we have the following

$$\Lambda' \subseteq \theta^{-1}(\theta(\Lambda')) = \theta^{-1}(Z_p) = \Lambda$$

and thus, $\Lambda' = \Lambda$, by the Riemann-Hurwitz formula. □

Lemma 2.13. *Let Δ be a Fuchsian group with the signature $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$, and let Λ be a Fuchsian group with the signature $(0; p, \overset{v}{\ldots}, p)$ for $v = sp^{n-1}+2$. Let $y_1, \ldots, y_s, w_1, w_2$, and $x_1, \ldots, x_v$ be sets of canonical generators of Δ and Λ respectively. Then, the mapping*

$$x_1, \ldots, x_v \mapsto \underbrace{y_1, \ldots, y_s}, \underbrace{y_1^{w_1}, \ldots, y_s^{w_1}}, \ldots, \underbrace{y_1^{w_1^{p^{n-1}-1}}, \ldots, y_s^{w_1^{p^{n-1}-1}}}, w_1^{p^{n-1}}, w_2^{p^{n-1}}$$

defines an embedding $\Lambda \hookrightarrow \Delta$.

Proof. It is sufficient to show that the product of images of $x_1, x_2, \ldots, x_v$ is trivial. But this is clearly the case. □

The following is an easy consequence of the previous two Lemmas.

Corollary 2.14. *Consider an action of $Z_{p^n} = \langle b \rangle$, defined by a $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$-generating vector:*

$$(b^{n_1 p^{n-1}}, \ldots, b^{n_s p^{n-1}}, b^{k_1}, b^{k_2}). \tag{16}$$

The corresponding $(0; p, \overset{v}{\ldots}, p)$-generating vector for the induced action of $Z_p = \langle a \rangle$ is equivalent to

$$(a^{n_1}, \overset{p^{n-1}}{\ldots}, a^{n_1}, \ldots, a^{n_s}, \overset{p^{n-1}}{\ldots}, a^{n_s}, a^{k_1}, a^{k_2}), \tag{17}$$

where $v = sp^{n-1} + 2$ and $a = b^{p^{n-1}}$.

Proof. Let Δ be a Fuchsian group with the signature $(0; p, \overset{s}{\ldots}, p, p^n, p^n)$ and let $\theta\colon \Delta \to Z_{p^n}$ be a smooth epimorphism defined by the generating vector (16). The subgroup $\Lambda = \theta^{-1}(Z_p)$ has the signature $(0; p, \overset{v}{\ldots}, p)$, by Lemma 1.4 and the Riemann-Hurwitz formula. Since the subgroup of Δ with this signature is unique by Lemma 2.12, the set

of generators from Lemma 2.13 is a canonical set of generators of Λ. Therefore, the generating vector for the action of Z_p is equivalent to (17) by Theorem 1.3. This concludes the proof. □

Theorem 2.15. *Let $n \geq 2$ be an integer and $p \geq 5$ be a prime. Then, for $g = p^{n-1}(p-1)/2$, the singular locus contains an ordinary connected component being the union of at least n equisymmetric strata defined by actions of p-groups. Moreover, if $n = 2$ the above g is the minimal genus for which $\mathcal{S}_g$ contains such a component.*

Proof. Consider a generating vector:

$$\tau = (a, \overset{p^{n-1}}{\ldots}, a, a^2, a^{-2})$$

defining an action of the group $Z_p = \langle a \rangle$ on a compact Riemann surface, which by the Riemann-Hurwitz formula has genus $g = p^{n-1}(p-1)/2$. This action does not extend to an action of a dihedral group D_p, since it has an odd number of fixed points. Moreover, by Lemmas 1.9, 1.10 and Theorem 1.3, it does not extend to an action of Z_{pq}, for $q \neq p$. Therefore, by Theorem 2.10, the stratum $\mathcal{M}_g^{Z_p,\tau}$ is not contained in $\mathcal{C}_g^{2,3}$, and so its Zariski closure is a connected component $\mathcal{C}$ of the singular locus.

We will show that this component contains at least n strata. For, consider an action of a group $Z_{p^n} = \langle b \rangle$ on a Riemann surface of genus g defined by a $(0; p, p^n, p^n)$-generating vector:

$$(b^{p^{n-1}}, b^2, b^{-2-p^{n-1}}).$$

By Lemma 2.14 and Theorem 1.3, this action extends the action of Z_p, so for every subgroup of Z_{p^n}, the corresponding equisymmetric stratum is contained in $\mathcal{C}$. The key observation here is that the induced action of any non-trivial subgroup of Z_{p^n} defines a nonempty stratum, since otherwise by Lemma 2.1 the stratum $\mathcal{M}_g^{Z_p,\tau}$ would be contained in $\mathcal{C}_g^{2,3}$. Furthermore, the stratum defined by the action of Z_{p^n} is Zariski-closed. Otherwise, it would be empty, since the Teichmüller dimension of its signature equals 0. This concludes the proof of the first part.

For the second part, suppose that $n = 2$, and let $\mathcal{C}$ be an ordinary connected component of $\mathcal{S}_g$ composed of at least two strata defined by actions of p-groups. Each such component contains a stratum defined by an action of Z_{p^2}, since otherwise $\mathcal{C}$ would be composed of only one stratum. Thus, by Theorem 2.5 the Riemann-Hurwitz formula $g = sp(p-1)/2$, for some integer s. This concludes the proof. □

Corollary 2.16. *The smallest g, for which $\mathcal{S}_g$ contains an ordinary connected component composed of more than one stratum, is equal to* 10. □

Remarks 2.17. In [6] authors claim that for genera up to 24, every connected component of the singular locus is either composed of only one stratum, or is equal to the exceptional component. The above corollary contradicts this assertion.

Remarks 2.18. A slightly stronger version of the above theorem was published in [37], where we asserted that, with the above assumptions, there is an ordinary connected component composed of exactly n strata. Working on this dissertation, we found a flaw in a proof of one of the related lemmas (Lemma 3.17). Unfortunately, it seems that the lemma itself is not correct, and neither is the theorem. Luckily, the only purpose of that lemma in [37] was to prove this theorem, so 2.15 fixes this bug to a significant extent.

3. Combinatorial properties of the exceptional component

The forthcoming section is devoted to the study of the Cornalba graph and its subgraphs. The main goal is to present the results regarding their diameters.

We start by setting some notational conventions. Throughout the section, we will denote by $\mathcal{G}_g(p)$ the subgraph of the Cornalba graph spanned by vertices corresponding to automorphisms of order p, these verticies will be denoted by v_i^p. The subscript will usually be omitted except for the case $i = 0$, by which we will denote the vertex corresponding to the action of Z_p with the minimal number of fixed points.

Recall that the *weight* of an edge connecting two vertices v^p and v^q is the number $|G|/pq$, where G is a group of the smallest order whose action extends the actions corresponding to v^p and v^q. An edge is said to be *bold* if and only if its weight is equal to 1. Let $\mathcal{H}$ be a subgraph of $\mathcal{G}_g$, by $\overline{\mathcal{H}}$ denote the subgraph spanned by all the vertices of $\mathcal{H}$ and all the vertices connected to some vertex of $\mathcal{H}$ by the bold edges.

Finally, the *reduced Cornalba graph* $\widehat{\mathcal{G}}_g$ is the subgraph of the Cornalba graph composed of all the vertices of $\mathcal{G}_g$ but only the bold edges.

Recall that by the results of Gromadzki and Wootton from [23], a spanning tree of $\mathcal{G}_g(p)$ is a star of radius 1 with the center v_0^p, which we will call the *center* of $\mathcal{G}_g(p)$.

Lemma 3.1. *The centers of $\mathcal{G}_g(p)$ and $\mathcal{G}_g(q)$ are connected directly i.e. by an edge.*

Proof. We will prove the assertion by showing that actions of Z_p and Z_q corresponding to v_0^p and v_0^q extend to the fixed points free action of Z_{pq}. Let $(h;-)$ and $(k;-)$ be signatures of actions of Z_p and Z_q respectively. By the Riemann-Hurwitz formula

$$q(k-1) = g - 1 = p(h-1),$$

thus q divides $h-1$, since $q \neq p$. Hence, there exists a positive integer a such that $2(g-1) = 2pq(a-1)$. Consider the action of $Z_{pq} = \langle b \rangle$ defined by the following $(a;-)$-generating vector $(b, \overset{2a}{\ldots}, b; -)$, which is the action we seek. □

The following characterization is a prompt consequence of Theorem 2.10.

Lemma 3.2. *An action defined by a smooth epimorphism $\theta\colon \Lambda \to Z_p$ defining a nonempty stratum corresponds to a vertex of the Cornalba graph if and only if it extends to an action of a group of order pq, where $q \neq p$ or to an action of Z_p^2.*

Proof. Recall that the vertex corresponding to the action of Z_p is contained in the Cornalba graph if and only if the stratum defined by this action is contained in the exceptional component. Therefore, the sufficiency follows directly from Lemmas 2.2, 2.3 and Theorem 2.5.

For the necessity, suppose that the stratum defined by the action of Z_p is contained in $\mathcal{C}_g^{2,3}$. First, by Lemma 1.7 we may assume that Z_p acts with the orbit genus $h < (p-3)/2$.

Therefore, the action of Z_p extends to an action of a group G of order pq for some prime number q, by Lemma 1.6. Suppose that $p = q$ and G is a cyclic group. Following the proof of Theorem 2.5, can assume that the signature of the action of G equals $(0; p, \overset{s}{\ldots}, p, p^2, p^2)$ since otherwise the action of Z_p extends to an action of Z_p^2 or to an action of a group of order $2p$. Therefore, the generating vector for the action of Z_p satisfies the assumptions of Theorem 2.10, by Lemma 1.10, which concludes the proof since the corresponding stratum is contained in the exceptional component. □

The following lemma allows us to see how the shortest paths connecting vertices isolated in $\mathcal{G}_g(p)$ with vertices of $\mathcal{G}_g(2)$ may look like.

Lemma 3.3. *Let $g \geq 5$ and suppose that v^p, where $p \geq 5$ is a vertex of the Cornalba graph isolated in $\mathcal{G}_g(p)$. Then, there exists a vertex v^2 for which (v^p, v^2) is a bold edge or v^q and v^2 for which (v^p, v^q, v^2) is a path in $\mathcal{G}_g$ such that the edges (v^p, v^q) and (v^q, v^2) are bold.*

Proof. First, assume that $g \neq p + 1$. Then, the fixed points free action of Z_p defines a nonempty stratum. By our assumption, the action of Z_p does not extend to an action of Z_p^2, since otherwise it would by connected by an edge with v_0^p by Theorem 3.10 in [23]. So, the action of Z_p extends to an action of a group G of order pq, where $p \neq q$ by Lemma 3.2. Furthermore, we may suppose that Z_p is normal in G. By Lemmas 2.2 and 2.3, the action of Z_p or Z_q extends to an action of a group of order $2p$. If the action of Z_q defines a nonempty stratum, then the proof is concluded.

So, suppose that the stratum corresponding to the action of Z_q is empty. Observe that $q > 3$ in such a case by Lemma 1.19. First, observe that the signature of the action of G is not on the Singerman's List, since the index of Z_q in G equals $p \geq 5$. We will show that in this case, the action of G also defines an empty stratum. For, assume to get a contradiction that $S = \mathcal{H}/\Gamma$ is a compact Riemann surface of genus g with the full automorphism group $\mathrm{Aut}(S) = \Lambda/\Gamma \cong G$. Moreover, let $Z_q \cong \Delta/\Gamma$. Since the action of Z_q defines an empty stratum, the group Δ is not finitely maximal. So there is a Fuchsian group Δ' with the signature appearing on the Singerman's List containing Δ and Γ as normal subgroups. Δ has to be normal in Δ' since none of the signatures from the second list of Singerman correspond to an action of Z_q on a surface of genus $g \geq 5$. Let $L = \Delta'/\Gamma$. Notice that the order of L is divisible by 2 or 3. Therefore, L is not a subgroup of G. Which is a contradiction, since we assumed that $G = \mathrm{Aut}(S)$. Hence, the action of G defines an empty stratum.

Again, following the proof of Lemma 2.1, the action of G is extended by an action of a group L' of order divisible by 2 or 3. Recall that by Lemma 1.9, the only periods in the signature of G are p, q, and pq. Therefore, G has to be normal in L', since none of the signatures on the second Singerman list from [36] is of the required form for such prime numbers. Clearly, Z_p is a characteristic subgroup of G, since it is unique by the Sylow

Theorems. Thus, it is normal in L. Finally, we have the paths (v^p, v^2) or (v^p, v^3, v^2), since every action of Z_3 extends to an action of a group of order 6. This concludes the proof in this case.

Now, suppose that $g = p + 1$. If $p > 5$, then every action of Z_p is fixed-point free, and so $\mathcal{G}_g(p)$ is empty. If $p < 5$, then $g < 5$, and this is not our case. Thus, we need only to consider the case $g = 6$. However, it is easy to check that every action of Z_5 on a compact Riemann surface of genus 6 extends to an action of a group of order 10 or to an action of Z_{15}. This concludes the proof. □

Since all edges constructed in the above lemma are bold, the reduced Cornalba graph is connected, and so there is a well-defined notion of a metric on $\widehat{\mathcal{G}}_g$. We are now in a position to prove the main theorem of this section.

Theorem 3.4. *The diameter of the Cornalba graph is at most* 5 *for all* $g \geq 5$.

Proof. Let v^p and v^q be arbitrary vertices of the Cornalba graph. We will construct a path connecting these vertices of length at most 5, dividing the proof into some independent cases.

Suppose first that v^p and v^q are not isolated in $\mathcal{G}_g(p)$ and $\mathcal{G}_g(q)$ respectively. Then, by Lemma 3.1, we have a path (v^p, v_0^p, v_0^q, v^q) of length 3.

Next assume that v^p lies on our spanning tree of $\mathcal{G}_g(p)$ and v^q is isolated in $\mathcal{G}_g(q)$. Then by Lemma 3.3 there is an edge (v^q, v^2) or a path $(v^q, v^{q'}, v^2)$ of length 2. But then

$$(v^p, v_0^p, v_0^2, v^2, v^q) \text{ or } (v^p, v_0^p, v_0^2, v_2, v^{q'}, v^q)$$

is a path in $\mathcal{G}_g$ and so the distance between v^p and v^q is at most 5.

Finally, suppose that v^p and v^q do not lie on our spanning trees of $\mathcal{G}_g(p)$ and $\mathcal{G}_g(q)$. Then, by Lemma 3.3 there are paths of length at most 2 connecting v^p and v^q with some 2-vertices v_1^2, v_2^2. If now (v_1^2, v_2^2) is an edge in $\mathcal{G}_g(2)$ then $d(v^p, v^q) \leq 5$. In the other case we can only deduce that $d(v^p, v^q) \leq 6$, since there is a path (v_1^2, v_0^2, v_2^2).

Observe that in the last case $d(v^p, v^q) = 6$ only if the path constructed above is of the form

$$(v^p, v^{p'}, v_1^2, v_0^2, v_2^2, v^{q'}, v^q),$$

where $p' \neq p$ and $q' \neq q$. If v^p or v^q is directly connected to a vertex v^2, then $d(v^p, v^q) < 6$. and we will show that if this is not the case then vertices $v^{p'}$ and $v^{q'}$ can be so chosen that the corresponding vertices v_1^2, v_2^2 are connected directly which will give a path $(v^p, v^{p'}, v_1^2, v_2^2, v^{q'}, v^q)$ is a path of length 5.

We will do so by showing that the actions of Z_2 corresponding to v_1^2 and v_2^2 have no more than $g + 3$ fixed points, which is sufficient by Theorem 3.14 in [23]. Observe that in our case, neither the action of Z_p nor of Z_q extends to an action of a group of even order.

Let $(h; p, \overset{r}{\ldots}, p)$ be the signature of the action of Z_p corresponding to v^p. Since v^p is not connected directly with $\mathcal{G}_g(2)$, $h < (p-3)/2$, by Lemma 1.7. There are two cases: the action of Z_p extends to an action of $Z_p \rtimes Z_{p'}$ or to an action of $Z_{pp'}$.

The first action corresponds to a signature $(k\,;\, p', \overset{s}{\ldots}, p', p, \overset{t}{\ldots}, p)$, where $t > 0$ by Lemma 3.1. Let h' be the orbit genus of the induced action of $Z_{p'}$. Since $Z_{p'}$ is not normal in $Z_p \rtimes Z_{p'}$, it coincides with its normalizer. So $h' \geq (p'-3)/2$, by Theorem 6 in [32]. By Lemma 1.17, the action of $Z_{p'}$ extends to an action of $Z_{2p'}$, where the induced action of Z_2 has at most $4p'-1$ fixed points.

If $h = 0$, then by Lemma 1.9, $s = 2$. Thus, the action of $Z_{p'}$ has 2 fixed points by Lemma 1.4, and so, by Lemma 1.16 we get what we want. So, suppose that $h \geq 1$ and assume first that $p \geq 11$. Then, by the Riemann-Hurwitz formula $p' \leq (g-1)/5$, and therefore, $4p'-1 \leq g+3$. For $p = 7$, $p' = 2$ or $p' = 3$, since p' divides $p-1$. But since v^p is not connected by and edge with a 2-vertex, $p' \neq 2$. If $t \geq 2$, then by the Riemann-Hurwitz formula $g \geq 19$ and so $4p'-1 \leq g+3$. If $t = 1$, then $r = 3$ since p' equals 3. But $r \leq 2h+2$, since $h \geq 1$, so the action of Z_7 extends to an action of Z_{14} by Lemma 1.14, which is not our case. For $p = 5$, the only possibility is $p' = 2$, which again is not our case.

The case when the action of Z_p extends to an action of $Z_{pp'}$ is more involved. Here, h' will also denote the orbit genus of the action of $Z_{p'}$. The signature of the action of $Z_{pp'}$ equals

$$(k\,;\, p', \overset{s}{\ldots}, p', p, \overset{t}{\ldots}, p, pp', \overset{u}{\ldots}, pp').$$

First, suppose that $h' \geq (p'-3)/2$. Assume that $h \geq 2$ and observe that $p \geq 11$ since $h < (p-3)/2$. For $t > 0$, by the Riemann-Hurwitz formula

$$2(g-1) = 2p(h-1) + r(p-1) \geq 2p + p'(p-1) \geq 10p' + 22,$$

since $r = tp' + u \geq p'$. This easily gives $p' \leq (g-12)/5$, and thus $4p'-1 \leq g+3$. For $t = 0$, $u = r$ and we can assume that $r > 6$ by Lemma 1.14. So, by the Riemann-Hurwitz formula,

$$2(h-1) + r\frac{p-1}{p} = 2p'(k-1) + s(p'-1) + r\frac{pp'-1}{p},$$

which easily gives $2h \geq 5(p'-1)$. Therefore, $p' \leq p/5$, since $h \leq (p-5)/2$. Now, observe that if $h \geq 2$, then $p \leq g-1$ by the Riemann-Hurwitz formula, and so,

$$4p'-1 \leq 4\frac{p}{5} - 1 \leq g+3.$$

Now let $h = 1$, by Lemma 1.9 the only case we need to consider is $k = 0$ and $s+u = p' = 3$. Observe that if $t = 0$, then $r \leq 3$, which is not our case by Lemma 1.14. So, by the Riemann-Hurwitz formula, we get that $2(g-1) \geq 3(p-1) \geq 18$, and thus $g \geq 10$, which ends the proof in this case.

Finally, let $h = 0$. By Lemma 1.9 $s+u = 2$. Thus $Z_{p'}$ acts with 2, $p+1$ or $2p$ fixed points. We only need to consider the latter two cases. Thus, the generating vector for the action of $Z_{p'} = \langle a \rangle$ is equivalent to

$$(1, \overset{2h'}{\ldots}, 1, a^p, \overset{p}{\ldots}, a^p, a^{-p^2}) \quad \text{or} \quad (1, \overset{2h'}{\ldots}, 1, a^p, \overset{p}{\ldots}, a^p, a^{-p}, \overset{p}{\ldots}, a^{-p}),$$

by Lemma 1.10. We will show that both actions extend to actions of a group $Z_{2p'} = \langle b \rangle$ such that the induced actions of Z_2 have at most $2p'+2$ fixed points. Let $u = (p-1)/2$. In the first case, consider a Fuchsian group Λ with the signature

$$(k'; 2, \overset{m}{\ldots}, 2, p, \overset{u}{\ldots}, p, 2p, 2p),$$

where integers k' and $m \in \{0, 2\}$ are such that $2h' = 4k' + m$. And a tuple

$$(1, \overset{2k'}{\ldots}, 1; b^{p'}, \overset{m}{\ldots}, b^{p'}, b^2, \overset{u}{\ldots}, b^2, b, b^{-p}),$$

to get the extension we are looking for. In the second case, consider the Fuchsian group with the signature

$$(k'; 2, \overset{m}{\ldots}, 2, p, \overset{u}{\ldots}, p, p, \overset{u}{\ldots}, p, 2p, 2p),$$

where k' and m are as above. And a tuple

$$(1, \overset{2k'}{\ldots}, 1; b^{p'}, \overset{m}{\ldots}, b^{p'}, b^2, \overset{u}{\ldots}, b^2, b^{-2}, \overset{u}{\ldots}, b^{-2}, b, b^{-1}),$$

to get the extension we needed. First, let $h' > 0$. By the Riemann-Hurwitz formula, we get that $p' \leq (g+2)/3$, since the number of fixed points of the action of $Z_{p'}$ is at least $p+1$ and $p \geq 5$. So, $2p'+2 \leq g+3$. Let $h' = 0$. Since $h' \geq (p-3)/2$ we get that $p' = 3$, and so $2p'+2 = 8 \leq g+3$, which ends the proof in this case.

For the case $h' < (p-3)/2$, we follow the proof of Lemma 2.3. We can assume that $p < p'$ since otherwise Z_p would extend to an action of a group Z_{2p}, which is not our case. First, notice that if the action of $Z_{p'}$ is fixed-points free, then there is nothing more to do since then $(v^{p'}, v_0^2)$ is an edge by Lemma 3.1. By the Riemann-Hurwitz formula

$$h = kp' + (s+u-2)(p'-1)/2,$$

which easily gives that $h = 0$, since $p < p'$ (cf. the proof of Lemma 2.3). and we can proceed as above. The only case that needs more consideration is $h' = 0$. But again, following the proof of Lemma 2.3, we can see that in this case, the action of Z_p extends to an action of Z_{2p}. This concludes the proof. □

Remarks 3.5. The cases $2 \leq g \leq 4$ are easy to deal with since we have the full classification of automorphism groups of compact Riemann surfaces of this genera [9], [10], [27]. Moreover, by [13], the Cornalba graphs for $g = 2, 3$ have less than 5 vertices, and using the classification of automorphism groups of compact Riemann surfaces of genus 3 we can see that the diameter of $\mathcal{N}_3$ equals 2.

In the case of genus 4 the Cornalba graph has 8 vertices: 3 of which are 2-vertices, 3 are 3-vertices and 2 are 5-vertices. It is easy to see that both of the 5-vertices are connected

by an edge with v_0^2 (cf. [9] and [27]). Moreover, every 3-vertex is connected by an edge with some 2-vertex. This immediately shows that the diameter of $\mathcal{N}_4$ is at most 4.

The above allows us to see the following immediately.

Corollary 3.6. *The diameter of the Cornalba graph is at most* 5 *for all* $g \geq 2$. □

Since every edge constructed above is bold, we can immediately see that the diameter of the reduced Cornalba graph is at most 5. The above results give hope for the calculation of the precise values of the diameters of the Cornalba graphs. However, it seems to be a highly challenging problem. So, now we focus on finding some lower bounds on the diameter of $\mathcal{G}_g$. We start with a technical lemma stated here for convenience, its proof is a straightforward checking, so we omit it.

Lemma 3.7. *Let* $p > 5$ *be a prime and* k *be an integer. If* $p > 7$*, then there exist integers* $2 < n < m < p$ *such that* $n + m \equiv k \pmod{p}$*. Moreover, if* $p = 7$*, such integers exists if and only if* $k \neq 5, 6 \pmod 7$. □

Theorem 3.8. *The diameter of the Cornalba graph is at least* 4 *for infinitely many genera.*

Proof. Let $g = 15k$, where $k \equiv 3 \pmod 5$ be an integer such that $g > 100$. Moreover, let $r_1 = (g+6)/3$ and $r_2 = (g+10)/5$. Observe that $r_1 \equiv 2 \pmod 5$ and $r_2 \equiv 1 \pmod 5$. We will construct vertices v and w corresponding to some 7-gonal and 11-gonal actions, for which $d(v, w) \geq 4$. We start by constructing actions of $Z_7 = \langle a \rangle$.

First, suppose that $r_1 \neq 5, 6 \pmod 7$. Consider a $(0; 7, \overset{r_1}{\ldots}, 7)$-generating vector

$$(a, \overset{r_1-7}{\ldots}, a, a^2, \overset{5}{\ldots}, a^2, a^n, a^m),$$

where n, m are integers from Lemma 3.7.

If $r_1 \equiv 5 \pmod 7$, then consider the following $(0; 7, \overset{r_1}{\ldots}, 7)$-generating vector:

$$(a, \overset{r_1-12}{\ldots}, a, a^2, \overset{5}{\ldots}, a^2, a^3, \overset{5}{\ldots}, a^3, a^4, a^6).$$

If $r_1 \equiv 6 \pmod 7$, then consider the following $(0; 7, \overset{r_1}{\ldots}, 7)$-generating vector:

$$(a, \overset{r_1-12}{\ldots}, a, a^2, \overset{5}{\ldots}, a^2, a^4, \overset{5}{\ldots}, a^4, a^5, a^6).$$

Defining an action of Z_7 on a surface of genus g.

For the action of $Z_{11} = \langle b \rangle$, we begin by assuming that $r_2 \neq 5, 6, 7, 8 \pmod{11}$. Consider a $(0, 11, \overset{r_2}{\ldots}, 11)$-generating vector:

$$(b, \overset{r_2-11}{\ldots}, b, b^2, \overset{5}{\ldots}, b^2, b^3, \overset{5}{\ldots}, b^3, b^a),$$

where a is uniquely determined integer, which by our assumption is not equal to $0, 1, 2, 3 \pmod{11}$.

In the remaining cases consider the following $(0, 11, \overset{r_2}{\ldots}, 11)$-generating vector:

$$(b, \overset{r_2-11}{\ldots}, b, b^2, \overset{5}{\ldots}, b^2, b^4, \overset{5}{\ldots}, b^4, b^a),$$

where again a is uniquely determined integer, which by our assumption is not equal to $0, 1, 2, 4 \pmod{11}$.

Let v and w be vertices of $\mathcal{G}_g$ corresponding to the actions of Z_7 and Z_{11}, respectively. Clearly, the action of Z_7 extends to an action of Z_{35} with the signature $(0; 7, \overset{u}{\ldots}, 7, 35, 35)$, where $u = (r_1 - 2)/5$. The induced action of $Z_5 = \langle c \rangle$ has the generating vector equivalent to $(1, \overset{2h}{\ldots}, 1; c, c^{-1})$, as shown in Lemma 1.10. We will now prove that the vertex corresponding to the action of Z_5 is the only vertex connected by an edge to v.

To this end, let G be an extension of the action of Z_7, giving an edge connecting v with v'. By the Costelnouvo-Severi inequality (3), the 7-gonal subgroup of G is unique and, hence, normal. If v' corresponds to an action of $Z_p \leq G$, then the subgroup $\langle Z_p, Z_7 \rangle$ has order $7p$. By Lemma 1.9 and Lemma 1.10, the action of Z_p must be equivalent to the action of Z_5.

A similar argument shows that w is connected by an edge only to the vertex corresponding to the action of Z_5 defined by the generating vector $(1, \overset{h'}{\ldots}, 1; , c, \overset{11}{\ldots}, c, c^{-1})$. However, since $g - 1$ is not divisible by 5, the vertices corresponding to these actions of Z_5 are not connected by an edge. Therefore, $d(v, w) \geq 4$, which completes the proof. □

Remarks 3.9. Observe that if g is even, then the vertices corresponding to the actions of Z_5 in the above proof are connected by edges with the vertex v_0^2, and so $d(v, w) = 4$ in this case.

Proposition 3.10. *Let $g \neq (5p - 1)/2$ for any prime number p. Then, the diameter of $\overline{\mathcal{G}_g(2)}$ is at most 3. Moreover, this bound is sharp for infinitely many genera.*

Proof. Let v and w be vertices of $\overline{\mathcal{G}_g(2)}$. The assertion is obvious if at least one of the vertices belongs to $\mathcal{G}_g(2)$. So, suppose that v, v^2 and w, w^2 are connected directly.

We will follow a similar tactic as in the proof of Theorem 3.4. We will show that v^2 can be chosen so that the corresponding action of Z_2 has the number of fixed points at most $g+3$. Recall that by Lemma 1.17 we can suppose that this number is smaller than $4p-1$.

Let $(h; p, \overset{r}{\ldots}, p)$ be the signature of the action of Z_p corresponding to v. By Lemma 1.16 we can assume that $r > 2$. First, suppose that $h > 2$ and $r > 3$. Then by the Riemann-Hurwitz formula $g - 1 \geq 4p - 2$ which easily gives want we need. If $r = 3$, then following the proof of Lemma 1.17 we can see that $u \leq 3$ and the number of fixed points of the corresponding action of Z_2 is at most $3p + 3$. So by the Riemann-Hurwitz formula $2(g - 1) \geq 7p - 3$, which again easily gives that $g + 3 \geq 3p + 3$.

Suppose that $h = 2$ and $r > 5$. Then by the Riemann-Hurwitz formula $2(g-1) \geq 8p-6$ and the assertion easily follows. Observe that $r \neq 3$, since otherwise, $g = (5p-1)/2$ by the Riemann-Hurwitz fomula. For the remainder cases, let

$$(1, 1, 1, 1; a^{n_1}, \ldots, a^{n_r})$$

be the generating vector for the action of $Z_p = \langle a \rangle$. By Theorem 1.3, we can assume that all n_i's are even. Consider a Fuchsian group Λ with the signature

$$(0; 2, \overset{m}{\ldots}, 2, 2p, \overset{r}{\ldots}, 2p),$$

where $m = 6 - r$, and a following tuple of elements of $Z_{2p} = \langle b \rangle$

$$(b^p, \overset{m}{\ldots}, b^p, a^{n_1+p}, \ldots, b^{n_r+p}),$$

which is clearly a generating vector for an action of Z_{2p} extending the action of Z_p. Moreover, the number of fixed points of the induced action of Z_2 equals $2p+2$ if $r = 4$ and $p+5$ if $r = 5$. By the Riemann-Hurwitz formula $2(g-1) = 2p + r(p-1)$, and the assertion easily follows. Now, suppose that $h = 1$. Let

$$(k; 2, \overset{m}{\ldots}, 2, p, \overset{t}{\ldots}, p, 2p, \overset{u}{\ldots}, 2p) \tag{18}$$

be the signature of an action of group G of order $2p$ extending the action of Z_p. If $k \neq 0$, then $m + u = 0$, by Lemma 1.9, and the assertion follows. So we can assume that $k = 0$ and $m + u = 4$. If G is not cyclic, then $u = 0$, $m = 4$, and by Lemma 1.4 the number of fixed points of an induced action of Z_2 equals 4, which ends the proof. So, suppose that G is cyclic. Observe that if $u = 0, 1$, then $t > 0$ since $r > 2$. So in these cases, the generating vector for the action of Z_p is equivalent to

$$(1, 1; a^{n_1}, a^{n_1}, \ldots, a^{n_t}, a^{n_t}) \text{ or } (1, 1; a^{n_1}, a^{n_1}, \ldots, a^{n_t}, a^{n_t}, a^k)$$

by Lemma 1.10. Recall, that we can assume that all of the n_i's and k are even. Consider a Fuchsian group Λ with the signature

$$(0; 2, \overset{m'}{\ldots}, 2, p, \overset{t-1}{\ldots}, p, 2p, \overset{v}{\ldots}, 2p),$$

where $v = u + 2$ and $m' = 4 - v$, and a tuple of elements of $Z_{2p} = \langle b \rangle$:

$$(b^p, \overset{m'}{\ldots}, b^p, (b^2)^{n_1}, \overset{t-1}{\ldots}, (b^2)^{n_{t-1}}, b^{n_t+p}, b^{n_t+p}),$$

when $u = 0$ and

$$(b^p, \overset{m'}{\ldots}, b^p, (b^2)^{n_1}, \overset{t-1}{\ldots}, (b^2)^{n_{t-1}}, b^{n_t+p}, b^{n_t+p}, b^{k+p}),$$

when $u = 1$. It is easy to see that the above tuples are generating vector defining actions that extend the action of Z_p, by Lemma 1.10. The above shows that we can assume that $u \geq 2$ and therefore, $m \leq 2$ by the Riemann-Hurwitz formula. Thus the induced action of Z_2 has $2p+2$, $p+3$ or 4 fixed points when $u = 2, 3, 4$ respectively. Recall that if $u = 2$, then $t \geq 1$ and thus $r \geq 4$. By the Riemann-Hurwitz formula $2(g-1) = r(p-1)$, which together with the above gives the assertion.

Finally, let $h = 0$ and let (18) again be the signature of an action of a group G extending the action of Z_p. By Lemma 1.9 $k = 0$ and $s + u = 2$. If G is not cyclic, then the number of fixed points of the induced action of Z_2 is 2 and the assertion follows.

Suppose that G is cyclic. First, observe that if $r = 3$, then by [28] and [13] the action of Z_p is either finitely maximal or defines an empty stratum, thus it does not correspond to a vertex of the Cornalba graph. So, we can suppose that $r \geq 4$. Using the same trick as above we can show, that we can assume $u > 0$. If $u = 2$, then the assertion follows, since then the number of fixed points of the induced action of Z_2 equals 2. If $u = 1$, then the action of Z_2 acts with $p + 1$ fixed points. Moreover, $r \geq 5$ since it has to be odd in this case. By the Riemann-Hurwitz formula $2(g-1) = -2p + r(p-1) \geq 3p - 5$, which easily gives the assertion. This ends the first part of the proof.

For the second part we will show that for every even $g > 16$ satisfying the following congruences

$$\begin{cases} g \equiv 2 \pmod 4 \\ g \equiv 1 \pmod 5, \end{cases} \tag{19}$$

there exists vertex v of $\overline{\mathcal{G}_g(2)}$, such that $d(v, u) = 3$, where u is the vertex corresponding to the hyperelliptic action of Z_2. The general idea of the proof is as follows. We will construct an action of Z_5 for which the corresponding vertex v is connected by an edge only with vertices of $\mathcal{G}_g(2)$ different than v_0^2. Since, the vertex corresponding to the hyperelliptic involution is connected only with v_0^2 out of vertices of $\mathcal{G}_g(2)$, $d(v, u) = 3$. For, let $r = (g+4)/2$ and consider the following action of $Z_5 = \langle a \rangle$

$$(a, \overset{r-3}{\ldots}, a, a^2, a^2, a^4). \tag{20}$$

Observe that by (19), the above is a well-defined $(0; 5, \overset{r}{\ldots}, 5)$-generating vector defining an action of Z_5 on a compact Riemann-surface of genus g. We will show that the vertex v, corresponding to (20), can only be connected by an edge to a vertex of $\mathcal{G}_g(2)$. For, assume that (v, w) is an edge, and w corresponds to an action of Z_p. Suppose that G extends the above-given action of Z_5 and the action of Z_p, by the Castelnuovo-Severi inequality (3), the 5-gonal subgroup of G is unique and thus normal. Therefore, the actions of Z_5 and Z_p are extended by an action of a group of order H $5p$. It is easy to see that H cannot be isomorphic to Z_5^2 nor to D_5 by Lemmas 1.11 and 1.12. So, H is cyclic, and it is obvious that $p = 2$ by Lemmas 1.10 and 1.9. Observe that, the generating vector (20) satisfies the assumptions of Lemma 1.14, and so the action of Z_5 extends to an action of Z_{10}. Thus, v is a vertex of $\overline{\mathcal{N}}_g(2)$. For the last part, let H be an extension of the action of Z_5 connecting v with some vertex w of $\mathcal{G}_g(2)$. We can assume that H has order 10. By Lemma 1.4 H is cyclic since, otherwise, r would be even, which is not our case. According to Lemma 1.9, the signature of the action of H has one period equal to 2, since otherwise the number of fixed points of the action of Z_5 would be even by Lemma 1.4. Thus $w \neq v_0^2$, which concludes the proof. $\square$

Remarks 3.11. The genera used in the second part of the above proof were chose such that it is easy to point the desired action of Z_5. However, it seems to be clear that we may define appropriate actions of other genera e.g. having different reminders while dividing by 5.

The following corollary provides an important clue in determining the diameter of the Cornalba graph.

Corollary 3.12. *Let $g \neq (5p-1)/2$ for any prime p and let v and w be vertices of $\mathcal{G}_g$, such that $d(v,w) = 5$. Then, v and w are not contained in $\overline{\mathcal{G}_g(2)}$.* □

As we mentioned earlier, the calculation of the precise value of the Cornalba graph is not an easy task in general. One of the main obstacles is the existence of non-bold edges, since there are no well-developed tools that allow to study such edges. Therefore, we now focus on the reduced Cornalba graph. Recall that the upper bound on the diameter of $\widehat{\mathcal{G}}_g$ equals 5.

Theorem 3.13. *The diameter of the reduced Cornalba graph is equal to 5 for infinitely many genera.*

Proof. Let us start with the observation that, by the Chinese Remainder Theorem and the Dirichlet Theorem on Primes in Arithmetic Progression, there are infinitely many primes satisfying the following three congruences

$$\begin{array}{lll} \text{(a)} & p \equiv & 4 \pmod 5 \\ \text{(b)} & p \equiv & 4 \pmod 9 \\ \text{(c)} & p \equiv & 120 \pmod{209} \end{array} \tag{21}$$

Given such p we shall construct for $g = 4p-2$ two vertices $v, w \in \widehat{\mathcal{G}}_g$ for which $d(v,w) = 5$. Observe that by (21a),

$$(a, \overset{p}{\ldots}, a, a^2, \overset{p}{\ldots}, a^2, a^3)$$

is a generating vector for a $(0; 5, \overset{2p+1}{\ldots}, 5)$-action of $Z_5 = \langle a \rangle$. Denote the corresponding vertex by v_5. Next by (21b),

$$(1, \overset{10}{\ldots}, 1; b, \overset{r-10}{\ldots}, 2, b, b^2, \overset{11}{\ldots}, b^2, b^3, \overset{11}{\ldots}, b^3, \ldots, b^{10}, \overset{11}{\ldots}, b^{10}, b^{12}, b^{14}, b^{18}),$$

where $r = (4p-79)/9$, is a $(5; 19, \overset{r}{\ldots}, 19)$-generating vector for an action of $Z_{19} = \langle b \rangle$ on a surface of genus $g = 4p-2$. Denote the corresponding vertex by v_{19}. We shall show that $d(v_5, v_{19}) = 5$.

First, observe that the action of Z_5 extends to a $(0; 5, 5, p, 5p)$-action of $Z_{5p} = \langle c \rangle$, defined by generating vector

$$(c^p, (c^p)^2, c^5, c^{2p-5}).$$

The induced action of Z_p has the signature $(2; p, \overset{6}{\ldots}, p)$ and, by Lemma 1.10, is defined by a the following generating vector

$$(1, 1, 1, 1; c^5, \overset{5}{\ldots}, c^5, (c^5)^{p-5}). \tag{22}$$

Denote the vertex obtained here by v_5.

In turn the action of Z_{19} extends to an action of $Z_{209} = \langle d \rangle$ defined by the generating vector

$$((d^{11}), \overset{(r-102)/11}{\ldots}, (d^{11}), (d^{11})^2, (d^{11})^3, \ldots, (d^{11})^{10}, d^{12}, d^{14}, d^{18}),$$

corresponding to signature $(0; 19, \overset{u}{\ldots}, 19, 209, 209, 209)$, where $u = (r-3)/11$. By our assumption on p, the above is a well-defined generating vector for the action of Z_{209}. The induced action of Z_{11} has the generating vector equivalent to

$$(1, \overset{2h}{\ldots}, 1; d^{19}, (d^{19})^3, (d^{19})^7),$$

where $h = (g-5)/11$ which is an integer since $11|4p-7$ by (21c). Denote the vertex obtained here by v_{11}.

Now, observe that by our assumption on p and the Castelnouvo-Severi inequality (3), the 5-gonal and $(19, 5)$-gonal actions are normal in their extensions. Using Lemmas 1.9, 1.10, and 1.11 it is easy to see that the signature of an extension of the action of Z_5 of order $5q$ has to be equal to $(0; 5, 5, p, 5p)$. But the induced action of Z_p is topologically equivalent to (22) by Lemma 1.10 and Theorem 1.3. So the degree of the vertex v equals 1, and it is connected with the vertex corresponding to the action of Z_p.

For the action of Z_{19} observe, that by Lemmas 1.9, 1.10 and 1.11 the signature of its extension of order $19q$ equals $(0; 19, \overset{u}{\ldots}, 19, 209, 209, 209)$, where $u = (r-3)/11$ which is an integer since $11|4p-7$ by (21c).

Thus the extension in question is Z_{209}. So, the induced action of Z_{11} has the signature of the form $(h; 11, 11, 11)$. Thus the corresponding generating vector is of the form

$$(1, \overset{2h}{\ldots}, 1; e^n, e^m, e^k), \tag{23}$$

where $e = d^{19}$. There are two possibilities either e^n, e^m, e^k are pairwise disjoint or (23) is equivalent to $(1, \overset{2h}{\ldots}, 1; e, e, e^9)$, by Theorem 1.3.

We will show that $d(v, w) > 4$. Let G be an extension of the action of Z_p of order pq. The orbit genus of Z_p is 2 so Z_p is normal in G by Theorem 6 in [32]. Moreover, by Lemma 1.11 G has to be cyclic. By Lemma 1.9 we can see that the only possibilities are $q = 2, 3, 5$. Thus $d(v, w) > 3$. First, suppose that $q = 5$. By Lemmas 1.4 and 1.9 and the Riemann-Hurwitz formula the signature of the action of G must be equal to

$$(0; 5, 5, p, 5p).$$

So the induced action of Z_5 has $2p+1$ fixed points. Suppose that H is a group of order 55 extending an action of Z_{11} with 3 fixed points. Observe that H is cyclic, since

otherwise the number of fixed points of Z_{11} would be divisible by 5. Moreover, by Lemma 1.10 the signature of the action of H is

$$(h'; 5, \overset{s}{\ldots}, 5, 55, 55, 55).$$

So, the number of fixed points of the induced action of Z_5 equals $11s + 3$ by Lemma 1.4. But, by our assumption $p \equiv 10 \pmod{11}$, thus $2p + 1 \equiv 10 \pmod{11}$. This means that $11s + 3 \neq 2p + 1$ and the induced actions of Z_5 are not topologically equivalent.

Now suppose that $q = 3$. By Lemma 1.10 the only possibility on the signature of G is

$$(0; 3, p, 3p, 3p, 3p).$$

By Lemma 1.4 the induced action of Z_3 has $p + 3$ fixed points.

On the other hand suppose that a group H of order 33, is an extension of an action of Z_{11} with 3 fixed points. Clearly, H is cyclic. Moreover, by Lemma 1.10 the signature of the action of H has to be of the form

$$(h'; 3, \overset{s}{\ldots}, 3, 33, 33, 33).$$

Thus the induced action of Z_3 has $11s+3$ fixed points. Since $p \neq 11s$, the induced actions of Z_3 are not topologically equivalent.

Now suppose that $q = 2$. Let, G be a group of order $2p$ extending the action of Z_p. Clearly, G is cyclic. Let

$$(k; 2, \overset{m}{\ldots}, 2, p, \overset{t}{\ldots}, p, 2p, \overset{u}{\ldots}, 2p),$$

be the signature of the action of G. The induced action of Z_2 has $mp + u$ fixed points. Moreover, by the Riemann-Hurwitz formula $6 = 4k + m + u$. Here the only possibilities are $u = 2, 4, 6$, and the induced action of Z_2 has $2, 6, 2p + 4, 4p + 2$ fixed points.

Let H be a group of order 22 extending tan action of Z_{11} with 3 fixed points. Again, by Lemma 1.11 H is cyclic. Let

$$(k'; 2, \overset{m'}{\ldots}, 2, 11, \overset{t'}{\ldots}, 11, 22, \overset{u'}{\ldots}, 22)$$

by the signature of the action of H. By Lemma 1.10 $t' = 0$ and $u' = 3$ or $t' = 1$ and $u' = 1$. Again by the Riemann-Hurwitz formula $2h + 2 = 4k' + m' + u'$, and the number of fixed points of the induced action of Z_2 has $11m' + 3$ or $11m' + 1$ fixed points respectively. But by our assumption $p \equiv 10 \pmod{11}$. Thus, $mp + u \neq 11m' + 3$ and $mp + u \neq 11m' + 1$ in all of our cases. This ends the proof. □

Remarks 3.14. The smallest integer 8689 satisfying (21) turns out to be prime. So since it is coprime with 9405 the sequence

$$a_k = 8689 + 9405k,$$

contains infinitely many primes satisfying (21), by Dirichlet theorem on primes in ar-itrhmetic progression.

Remarks 3.15. The assumptions on p in the above Theorem are taken so that we can easily construct suitable vertices. But it seems to be clear, that these assumptions are to strong, and the way we constructed vertices v and w is not the only possible.

Remarks 3.16. Theorem 3.13 shows that we cannot strengthen the upper bound on the diameter of the Cornalba graph from Theorem 3.4 using methods based on constructing bold edges.

4. Properties of connected components composed of only one stratum

The forthcoming section is devoted to studying the properties of finitely maximal actions of groups of prime orders. Recall that a group of automorphisms of a compact Riemann surface of genus g is said to be *finitely maximal*, if viewed as a subgroup of the mapping class group $\mathfrak{M}_g$, it has no finite extensions there. Recall also that strata defined by finitely maximal actions of Z_p are connected components of the singular locus, underscoring their importance and interest in our dissertation.

4.1. **Existence of finitely maximal action with given branch data.** We start this subsection by introducing the notation. Let

$$(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_r), \tag{24}$$

be a generating vector for an action of Z_p. By theorem 1.3, a permutation of $a_1, \ldots, a_r$ in (24) gives rise to an equivalent action. Therefore, without loss of generality, given a prime q, we can assume that our action is defined by

$$(1, \overset{2h}{\ldots}, 1; b_1, \overset{qn_1}{\ldots}, b_1, \ldots, b_t, \overset{qn_t}{\ldots}, b_t, c_1, \ldots, c_u),$$

for some $u \geq 0$, $n_1, \ldots, n_t > 0$ such that

- b_i are pairwise distinct,
- each c_i appears less than q times,
- $u + (n_1 + \ldots + n_t)q = r$

called a *q-form* of the vector (24). The sequences $(b_1, \overset{qn_1}{\ldots}, b_1, \ldots, b_t, \overset{qn_t}{\ldots}, b_t)$ and $(c_1, \ldots, c_u)$ will be called the *q-block* and *q-tail* of the vector (24) respectively. Furthermore, the sequences $(b_i, \overset{qn_i}{\ldots}, b_i)$ will be called the *q-subblocks of multiplicity* n_i. This observation is crucial and we will use it extensively throughout the section. Recall that by a nontrivial action of Z_q on Z_p we mean an action defined by a nontrivial homomorphism $Z_q \to \mathrm{Aut}(Z_p)$. The following lemma, together with Lemma 1.14, will become very useful in this section.

Lemma 4.1. *Let $(1, \overset{2h}{\ldots}, 1; a_1, \ldots, a_r)$ be a generating vector for an action of Z_p whose nontrivial coordinates can be decomposed into orbits of the nontrivial action of Z_2 on Z_p. Then, the action of Z_p extends to an action of D_p.*

Proof. By our assumption and Theorem 1.3 the generating vector for the action of $Z_p = \langle a \rangle$ is equivalent to

$$(1, \overset{2h}{\ldots}, 1; a_{i_1}, a_{i_1}^{-1}, \ldots, a_{i_k}, a_{i_k}^{-1}),$$

where $k = r/2$. Consider a $(0; 2, \overset{s}{\ldots}, 2, p, \overset{k}{\ldots}, p)$-generating vector for an action of $D_p = \langle a, b | a^p, b^2, bab^{-1}a \rangle$

$$(b, \overset{s-1}{\ldots}, b, ba^{-n}, a_{i_1}, a_{i_2}, \ldots, a_{i_k}), \tag{25}$$

where $s = 2h+2$ and $a_{i_1} \ldots a_{i_k} = a^n$. The action of D_p defined by (25) extends the action of Z_p by Lemma 1.11. $\square$

Our goal is to fully characterize the signatures that can correspond to finitely maximal actions of Z_p. We begin by recalling the main result of Peterson, Russel, and Wootton from [32].

Theorem 4.2. *If $h \leq (p-5)/2$ and $r > p+7$, then there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$.* □

We are in a place to state the main theorem of this section. For the sake of readability, we will divide the proof of the following into theorems, lemmas, and remarks since it requires considering multiple cases.

Theorem 4.3. *There exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$ if and only if one of the following conditions holds*

- *$h \leq (p-9)/2$ and $r \geq 2h+3$,*
- *$h = (p-7)/2$, $p > 7$ and $r = 2h+3$ or $r \geq 2h+5$,*
- *$h = (p-5)/2$, $p \geq 7$ and $r = p, p+2$ or $r \geq 2h+9$,*
- *$h = 0$, $p = 7$ and $r \geq 5$,*
- *$h = 0$, $p = 5$ and $r = 7$ or $r \geq 9$.*

□

We start with a theorem similar to Theorem 4.2.

Theorem 4.4. *Let $p \geq 7$ be a prime, $h \leq (p-7)/2$ and $r \geq 2h+5$ be integers. Then, there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$.*

Proof. Let $Z_p = \langle a \rangle$. First, observe that a generating vector of the form

$$(1, \overset{2h}{\ldots}, 1; a, \overset{r-2h-3}{\ldots}, a, a^{n_1}, \ldots, a^{n_{2h+3}}),$$

where a^{n_i} are pairwise disjoint nontrivial elements of $Z_p \setminus \{a\}$, defines a finitely maximal action of Z_p. To see this note that such an action cannot extend to an action of Z_p^2, since by Lemma 1.12 every element would occur at least p times in the generating vector. Moreover, it cannot extend to an action Z_{pq}, since by Lemmas 1.10 and 1.9, there would be at most $2h+2$ elements that occur only once in the generating vector. Finally, such an action cannot extend to an action of a nonabelian semidirect product $Z_p \rtimes Z_q$, since the generating vector does not decompose into orbits of the action of Z_q on Z_p.

To construct a generating vector for an action of Z_p of the above form, we need to find a collection $n_1, \ldots, n_{2h+3}$ of nontrivial pairwise disjoint elements of $\mathbb{Z}_p^*$ such that

$$n_1 + \ldots + n_{2h+3} \equiv 2h + 3 - r \pmod{p}.$$

Consider an equation $x + y \equiv r - 2h - 4 \pmod{p}$. It is easy to check that this equation has a solution such that $x \neq y$ and $x, y \notin \{0, 1\}$. So, the set $\mathbb{Z}_p \setminus \{0, 1, x, y\}$ consists of $p-4$ elements such that the sum of them is equal to $2h+3-r \pmod{p}$. Moreover, if

k is an element of this set different from $-1, -x, -y$, then $-k$ is also an element of this set. So we can remove as many pairs $(k, -k)$ from the set to obtain a collection we are looking for. □

Remarks 4.5. It is easy to check that there are no finitely maximal actions of Z_7 with the orbit genus 0 having fewer than five fixed points. Therefore, the above theorem fully describes signatures of finitely maximal 7-gonal actions.

Lemma 4.6. *Let r be an integer such that $r > 3$ if it is odd and $r > 4$ if it is even. Let k be the number of prime divisors of r. Then $2k \leq r - 3$ if r is odd and $2k \leq r - 2$ if r is even.*

Proof. First, suppose that r is odd. Thus, $r \geq 5$. If $k = 1$, then the assertion is obvious. So, assume that $k \geq 2$. Observe that $r = p_1^{a_1} \dots p_k^{a_k} \geq 3^k$, since every $p_i \geq 3$. Therefore, it is sufficient to prove that $2k \leq 3^k - 3$, which is clearly true for every $k \geq 2$.

Now, suppose that r is even. Again, it is easy to see that the assertion is true when $k = 1$, since then $r \geq 8$. So, assume that $k \geq 2$. Similarly as above, observe that $r \geq 2 \cdot 3^{k-1}$. Thus, we need to show that $2k \leq 2 \cdot 3^{k-1} - 2$, which again is clearly true for $k \geq 2$. □

Theorem 4.7. *Let $p > 7$ be a prime, and let $h \leq (p-9)/2$, and $r \geq 2h + 3$ be integers. Then, there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\dots}, p)$.*

Proof. Observe that by Theorem 4.4 we only need to consider two cases $r = 2h+3, 2h+4$. The proofs in both of these cases are similar; thus, we will consider only the first in full detail. We start by outlining our strategy to construct a finitely maximal action of $Z_p = \langle a \rangle$ with the given signature. Observe that by Lemmas 1.10, 1.12, and 1.9 an action of Z_p defined by a generating vector $(1, \overset{2h}{\dots}, 1; a^{n_1}, a^{n_2}, \dots, a^{n_r})$, where elements a^{n_i} are pairwise distinct, does not extend to an action of Z_{pq} or Z_p^2. Therefore, we need to ensure that the vector $(a^{n_1}, a^{n_2}, \dots, a^{n_r})$ cannot be decomposed into orbits of a nontrivial action of Z_q on Z_p. We will do so by ensuring that for any nontrivial action of Z_q on Z_p, such that q divides r, the vector contains an element of the orbit of the generator a, but does not contain the orbit entirely. Clearly, such a generating vector satisfies the assumptions of Corollary 1.13.

So, suppose that $r = 2h + 3$. We will find nontrivial elements $n_1, \dots, n_r$ of the ring $\mathbb{Z}_p$ that meet our goal. First, suppose that $r = 3$ and consider the following generating vector

$$(a, a^2, a^{p-3}),$$

which is a vector we are looking for, since $p > 7$. Now, suppose that $r > 3$ and consider the multiplicative group $\mathbb{Z}_p^*$ of the ring $\mathbb{Z}_p$. Let $S = \{\lambda_1, \dots, \lambda_n\}$ be a set of generators of all subgroups of prime orders of $\mathbb{Z}_p^*$ whose orders divide r. We may choose these generators

so that 2 and $p-3$ are not elements of this set, since the only case in which we would not be able to do so is when one of these elements has the order equal to two, or they are contained in the subgroup of order three, which is clearly not the case. It is important to notice that if $\lambda \in S$, then $-\lambda \notin S$. Let

$$n_1 = -1,\ n_2 = -2,\ \text{and } n_3 = 3.$$

Now we will find the remaining $2h$ elements. Let $S_1 = \{\lambda_{i_1}, \ldots, \lambda_{i_m}\} = S \setminus \{-1, -2, 3\}$. Observe that the set $\mathbb{Z}_p^* \setminus \{1, 2, p-3, -1, -2, 3\}$ can be decomposed into sets $S_1, -S_1$ and B. Notice that $x \in B$ if and only if $-x \in B$. Let k be the number of prime divisors of r. Since $r > 3$, we have $2k \leq r-3$ by Lemma 4.6. Furthermore, we have $2m \leq 2n \leq 2k \leq r-3 = 2h$, since n is the number of different prime-ordered subgroups of $\mathbb{Z}_p^*$ whose orders divide r. So, we can choose the next $2m$ elements as

$$n_4 = \lambda_{i_1}, \ldots, n_{3+m} = \lambda_{i_m}, n_{4+m} = -\lambda_{i_1}, \ldots, n_{3+2m} = -\lambda_{i_m}.$$

The remaining elements can be chosen from the set B by taking suitable many pairs $x_i, -x_i$.

In the case $r = 2h+4$, we may follow the above construction starting with a set $S = \{\lambda_1, \ldots, \lambda_n\}$ of generators of prime ordered subgroups of $\mathbb{Z}_p^*$ whose orders divide r, such that the numbers $1, 2, 3, p-6$ are not elements of this set. The only part that needs more consideration is the use of Lemma 4.6 to estimate the cardinality of the set S_1. However, observe that in this case $\mathbb{Z}_p^*$ has the subgroup of order 2, which is generated by the element -1. Which means that $-1 \in S$. Therefore, $S_1 = S \setminus \{-1, -2, -3, 6\}$ would have at least one element less than S. Thus, $2m \leq 2(n-1) \leq 2(k-1)$, which by Lemma 4.6, gives us that $2m \leq r-4 = 2h$. This concludes the proof. □

The above construction may seem a bit intricate, so we present an example to illustrate our proof.

Example 4.8. Consider a signature $(10; 31, \overset{24}{\ldots}, 31)$. We will construct a finitely maximal action of Z_{31} with this signature following our construction. There are two prime ordered subgroups of $\mathbb{Z}_{31}^*$ whose orders divide the number of fixed points i.e. the subgroups of orders 2 and 3. Let $S = \{5, 30\}$ be a set of generators of these groups. Then the set $\mathbb{Z}_p^* \setminus \{1, 2, 3, p-6, -1, -2, -3, 6\}$ can be decoposed into sets $S_1 = \{5\}$, $-S_1 = \{26\}$ and $B = \{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27\}$. So, we take the following

$$n_1 = 30, n_2 = 29, n_3 = 28, n_4 = 6, n_5 = 5, n_6 = 26.$$

Now, we need to choose eighteen elements from the set B. However, it is clear that we may simply take suitable pairs $x, -x$ from the set B.

Lemma 4.9. *Suppose that Z_p, where $p \geq 7$ acts with the signature $(h; p, \overset{r}{\ldots}, p)$, such that $h = (p-7)/2$ and $r = 2h+4$. Then, the action of Z_p is not finitely maximal.*

Proof. Let

$$(1, \overset{2h}{\ldots}, 1; a_1, \ldots, a_r) \tag{26}$$

be a generating vector for the action of Z_p. There are two possible cases. First, there are indexes $i \neq j$ such that $a_i = a_j$. In this case, the 2-tail of the vector (26) has the length $u \leq r - 2 = 2h + 2$. Thus, the action of Z_p is not finitely maximal by Lemma 1.14.

Second, elements $a_1, \ldots, a_r$ are pairwise disjoint. In this case the set of all nontrivial elements of Z_p can be decomposed into two subsets $\{a_1, \ldots, a_r\}$ and $\{x, y\}$, since $r = p - 3$. The product of all elements of Z_p is trivial. Moreover, the product $a_1 \ldots a_r$ is also trivial by the definition of a generating vector. Hence $x = y^{-1}$ and the set $a_1, \ldots, a_r$ can be decomposed into orbits of the nontrivial action of Z_2 on Z_p. Thus, the action of Z_p extends to an action of D_p by Lemma 4.1. □

Lemma 4.10. *Let $p > 7$ be a prime, $h = (p-7)/2$ and $r = 2h + 3$. Then, there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$.*

Proof. We will construct a generating vector for an action of Z_p that satisfies the assumptions of Corollary 1.13. First, observe that for these assumptions to be satisfied, we only need to ensure that the vector is composed of pairwise different elements and cannot be decomposed into orbits of any nontrivial action of Z_q on Z_p. Recall that when an action of Z_p extends to an action of a non-abelian semidirect product $Z_p \rtimes Z_q$, then the number of its fixed points is divisible by q. Thus, in our case, the only possibility is $q = 3$, since the greatest common divisor of $p - 1$ and $2h + 3 = p - 4$ equals 1 or 3. So, first suppose that $p - 1$ is not divisible by 3 and consider the vector

$$(1, \overset{2h}{\ldots}, 1; a, a^4, a^5, \ldots, a^{p-6}, a^{p-4}, \ldots, a^{p-1}),$$

Now, suppose that $p - 1$ is divisible by 3 and let $1 \leq k \leq p - 1$ be a number such that $k^3 \equiv 1 \pmod{p}$. Observe that $k > 2$ since $p > 7$. Thus, we may remove elements $k, -1, 1 - k$ from the set of elements of $\mathbb{Z}_p^*$ to obtain nontrivial elements $n_1, n_2, \ldots, n_r$ of the ring $\mathbb{Z}_p$ having a trivial sum. It is important to notice that $1 = n_i$, for some i. Then, the vector $(1, \overset{2h}{\ldots}, 1; a^{n_1}, \ldots, a^{n_r})$, is the one we seek. To see this, observe that the vector $(a^{n_1}, \ldots, a^{n_r})$ cannot be decomposed into orbits of a nontrivial action of Z_3 on Z_p, since a is an elements of this sequence and a^k is not, thus the sequence does not contain the orbit of a entirely. □

Lemma 4.11. *Let $p \geq 7$ be a prime, $h = (p-5)/2$ and $r \geq 2h + 3$ be integers. Then, there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$ if and only if $r = p, p + 2$ or $r \geq 2h + 9$.*

Proof. By Theorem 4.2, we only need to consider the cases $2h + 3 = p - 2 \leq r \leq p + 7$. We begin by examining the cases in which the signature $(h; p, \overset{r}{\ldots}, p)$ does not

correspond to a finitely maximal action. First, we will show that an action of Z_p with $p-2$ fixed points and orbit genus h cannot be finitely maximal. To see this, suppose that $(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_{p-2})$ is a generating vector for an action of Z_p. The product of all nontrivial elements of Z_p is trivial, so the product of $p-2$ pairwise distinct elements of Z_p cannot be trivial. Therefore, there are $i \neq j$ such that $a_i = a_j$. But then, the 2-tail of this vector has length $u < p-3 = 2h+2$. This means that such an action of Z_p extends to an action of Z_{2p} by Lemma 1.14. Analogously, there is no finitely maximal action of Z_p with orbit genus h and $p-1$ fixed points. Indeed, the only vector of $p-1$ pairwise different nontrivial elements of Z_p consists of all nontrivial elements of Z_p, thus it satisfies the assumptions of Lemma 4.1 and extends to an action of D_p.

Now, we will show that there are no finitely maximal actions with orbit genus h and $p+1$ fixed points. For, let $(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_{p+1})$, be a generating vector for an action of Z_p. Clearly, the elements $a_1, \ldots, a_r$ cannot be pairwise distinct. Without loss of generality, we may assume that $a_1 = a_2$. Observe that if elements $(a_3, \ldots, a_{p+1})$ are pairwise different, then this sequence contains all nontrivial elements of Z_p. Therefore, the generating vector for the action of Z_p is equivalent to $(1, \overset{2h}{\ldots}, 1; a_1, a_1, a, a^2, a^3, \ldots, a^{p-1})$, which is not a well-defined generating vector. This contradiction means that there are elements in $(a_3, \ldots, a_{p+1})$, which occur more than once in this sequence. Thus, the length of the 2-tail does not exceed $p-3 = 2h+2$, and the assertion follows from Lemma 1.14.

The case $r = p+3$ is similar, so we will not consider it in full detail. Let $(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_{p+3})$ be a generating vector for an action of Z_p. Observe that, without loss of generality, we may assume that $a_1 = a_2$ and $a_3 = a_4$. So, the vector takes the form $(1, \overset{2h}{\ldots}, 1; a_1, a_1, a_3, a_3, a_5, \ldots, a_{p+3})$. Therefore, its 2-tail has length at most $p-1$. Clearly, if the elements $a_5, \ldots, a_{p+3}$ are not pairwise different, then the length of the 2-tail does not exceed $p-3 = 2h+2$. So, the action of Z_p extends to an action of Z_{2p} by Lemma 1.14. Now, if the sequence $(a_5, \ldots, a_{p+3})$ consists of pairwise distinct elements, then it contains all nontrivial elements of Z_p. Thus, by the definition of the generating vector, $a_1^2 a_3^2 = 1$, hence $a_1 = a_3^{-1}$. This means that the generating vector satisfies the assertions of Lemma 4.1, so the action of Z_p extends to an action of D_p.

Now suppose that $p \leq r \leq p+7$. First, suppose that $r-p+2$ is even. Let $n_1, \ldots, n_{p-2}$ be the elements of the set $\mathbb{Z}_p^* \setminus \{r-p+2\}$. Consider the following generating vector for the action of $Z_p = \langle a \rangle$

$$(1, \overset{2h}{\ldots}, 1; a, \overset{r-p+2}{\ldots}, a, a^{n_1}, \ldots, a^{n_{p-2}}). \tag{27}$$

We will show that the action of Z_p defined above is finitely maximal. Observe that this action cannot extend to an action of Z_p^2, nor to an action of a nonabelian semidirect product $Z_p \rtimes Z_q$ for any prime number q, since the only nontrivial element that occurs more than once in the vector (27) is a. Next, we show that the action of Z_p cannot extend to an action of a cyclic group Z_{pq}. For any $q > 2$, the q-tail of the vector (27) has length $u \geq p-3 > h+2$, by our assumption on h and since there are at least $p-3$ elements that

occur only once in the generating vector, which contradicts Lemma 1.9. Furthermore, since $r-p+2$ is even, the 2-tail of this vector has exactly the length $p-2=2h+3$, which again contradicts Lemma 1.9.

To conclude the proof, we need to consider the cases $r=p+5$ and $r=p+7$. The proofs in both cases are analogous; thus, we will present only the first in full detail. Suppose that $r=p+5$, and consider the vector

$$(1,\overset{2h}{\ldots},1;a,a,a,a,a^{-2},a^{-2},a^{-2},a,a^2,,\ldots,a^{p-3},a^{p-1}).$$

First, observe that this is indeed a well-defined generating vector. Next, note that the action of Z_p cannot extend to an action of Z_p^2, nor to an action of a nonabelian semidirect product. Furthermore, any q-tail has length at least $p-1=2h+4$, which concludes the proof by Lemma 1.9. For the case $r=p+7$ consider the vector

$$(1,\overset{2h}{\ldots},1;a,\overset{6}{\ldots},a,a^{-3},a^{-3},a^{-3},a,\ldots,a^{p-4},a^{p-2},a^{p-1}).$$

We can use similar arguments as above to see that the action of Z_p defined by this vector is finitely maximal. □

Lemma 4.12. *Let $r>2$ be an integer. There exists a finitely maximal action of Z_5 with the signature $(0;5,\overset{r}{\ldots},5)$ if and only if $r=7$ or $r\geq 9$.*

Proof. We only need to consider the cases $3\leq r\leq 12$ by Theorem 4.2. It is easy to see that when $r\leq 6$ or $r=8$, there are no finitely maximal actions of Z_5 with r fixed points. When $r=7,9,10,12$ consider the following generating vector

$$(a,a,a,a^3,a^3,a^2,a^4),(a,\overset{6}{\ldots},a,a^2,a^3,a^4),(a,\overset{5}{\ldots},a,a^3,a^3,a^3,a^2,a^4),(a,\overset{7}{\ldots},a,a^2,a^2,a^2,a^3,a^4)$$

respectively. Each of the above vectors defines a finitely maximal action of Z_5. This concludes the proof. □

The above theorems and lemmas cover all possible cases of the signature $(h;p,\overset{r}{\ldots},p)$ that needed to be considered, and therefore the proof of Theorem 4.3 is concluded.

4.2. **Combinatorial density of the set of finitely maximal actions.** In this subsection, we will study the finitely maximal action from the quantative point of view. The results of this subsection were published in [20]. We start this section by introducing some notation. Recall that a subset Y of a countable set X is said to be combinatorially dense with respect to a filtration $X_1\subseteq X_2\subseteq\ldots\subseteq X_n\subseteq\ldots$ if and only if

$$\lim_{n\to\infty}\frac{|X_n\cap Y|}{|X_n|}=1.$$

Let $\Lambda_h^r(p)=\Lambda$ denote a Fuchsian group with the signature $\sigma=\sigma_h^r(p)=(h;p,\overset{r}{\ldots},p)$. Recall that the Teichmüller dimension of Λ is equal to $d=3(h-1)+r$. Let $D=D_d$ be

the set of pairs (h, r) of integers that satisfies the following two conditions:

$$h \in [0, d/3 + 1] \text{ and } r = d - 3(h - 1).$$

The set of all classes of actions of groups Z_p of dimension d will be denoted by A^d, and its subset composed of finitely-maximal actions will be denoted by M^d.

As we mentioned in the Subsection 1.6, we may understand the limit (9) as the proportion of the subset Y and the set X with respect to a filtration. Here we want to study the proportion of the set of finitely maximal actions of Z_p of given dimension in the set of all actions of this dimension.

There are at least two natural approaches one can take to look at our problem: the geometric and the algebraic. The first one corresponds to the filtration G_g of A^d, where G_g is the set of all actions from A^d that correspond to automorphism groups of compact Riemann surfaces of genus at most g. We will not consider this filtration here; however, the study of the limit (9) will be an interesting direction of future research.

In the second case we consider the following filtration. First, let $\{p_1, p_2, \ldots, p_n, \ldots\}$ be the set of all prime numbers ordered increasingly. Let $A_h^r(p)$ be the set of all classes of actions of Z_p with the signature $\sigma_h^r(p)$, and let

$$A(p) = \bigcup_{(h,r)\in D_d} A_h^r(p).$$

Finally let

$$A_k = \bigcup_{i=1}^{k} A(p_i). \tag{28}$$

The main goal of this section is to prove that the set M^d is combinatorially dense in the set A^d with respect to the filtration (28). Firstly, let $a_h^r(p)$ (respectively $a(p)$, a_k) be the cardinality of $A_h^r(p)$ (respectively $A(p)$, A_k).

Theorem 4.13. *The set M^d is combinatorially dense in the set A^d of all classes of actions of prime orders of the Teichmüller dimension d with respect to the filtration* (28).

Proof. Throughout the proof we will omit the parameter d, since it is fixed. We need to show that

$$\lim_{k\to\infty} \frac{m_k}{a_k} = 1,$$

where m_k is the cardinality of $M \cap A_k$. However, we find it convenient to prove that

$$\lim_{k\to\infty} \frac{e_k}{a_k} = 0,$$

where $e_k = |E \cap A_k|$, for $E = A \setminus M$.

Let $e_h^r(p)$, $e(p)$, be the cardinality of sets $E \cap A_h^r(p)$ and $E \cap A(p)$ respectively. We will find sufficient upper and lower bounds for e_k and a_k. We start with the estimation of a_k.

For a given pair $(h,r) \in D_d$ let $C_h^r(p)$ be the set of all sequences $(b_1,\ldots,b_r)$ of non-trivial elements of $Z_p = \langle b \rangle$ with the trivial product. Let $c_h^r(p)$ be its cardinality. First, observe that if $r = 0$, there is only one action of Z_p. Thus, we may consider only the cases where $r > 0$. Since an action of Z_p cannot have only one fixed point, we may assume that $r \geq 2$. Observe that the first $r-2$ elements of a sequence $b_1,\ldots,b_r$ can be chosen arbitrarily, thus there are $(p-1)^{r-2}$ of them. Furthermore, the remaining elements can be chosen in $p-1$ ways when $b_1 \ldots b_{r-2} = 1$ and in $p-2$ otherwise. Finally, we get that

$$(p-1)^{r-2}(p-2) \;\leq\; c_h^r(p) \;\leq\; (p-1)^{r-1}. \tag{29}$$

The classes of actions of Z_p with the signature $\sigma_h^r(p)$ are in bijective correspondence with the classes of sequences from $C_h^r(p)$ regarding the relation given in Theorem 1.3. This relation is defined by permuting a sequence and applying an automorphism of Z_p to it. Hence, to obtain a lower bound we seek, we may simply divide the left-hand side of the first inequality of (29) by $r!p$, since $p-1 < p$. By doing so we get the following.

$$a(p) \geq a_0^{d+3}(p) \geq \frac{(p-1)^{d+1}(p-2)}{p(d+3)!}$$

and finally

$$a_k = \sum_{i=1}^{k} a(p_i) \geq \sum_{i=1}^{k} \frac{(p_i-1)^{d+1}(p_i-2)}{p_i(d+3)!} \geq \frac{(p_k-1)^{d+1}(p_k-2)}{p_k(d+3)!} \tag{30}$$

Now, we will look for a suitable upper bound on e_k. First, observe that for $h > 0$ we have the following by the second inequality of (29)

$$e_h^{d-3(h-1)}(p) \leq (p-1)^{d-1}. \tag{31}$$

This is also true when $r = d-3(h-1) = 0$, since such a case may occur only when $d > 2$. As it turns out, the above will be sufficient for our purposes. We now focus on the case $h = 0$. Observe that by case (i) of Lemma 1.9, an extension of an action of Z_p with the orbit genus 0 has one of the following signatures.

$$\begin{array}{lll} (1) & (0; q, q, p, \overset{t}{\ldots}, p) & Z_p \rtimes Z_q = \langle b, c \rangle, \text{ where } t = r/q, \\ (2) & (0; q, p, \overset{t}{\ldots}, p, pq) & Z_{pq} = \langle d \rangle, \text{ where } t = (r-1)/q, \\ (3) & (0; p, \overset{t}{\ldots}, p, pq, pq) & Z_{pq} = \langle d \rangle, \text{ where } t = (r-2)/q. \end{array} \tag{32}$$

The set $E_0^r(p)$ is the union of the sets $E_0^{r,i}(p)$ composed of classes of actions of Z_p with the signature $\sigma_0^r(p)$ extending to some action corresponding to the above three cases. Let $E_+(p)$ be a set composed of all extendable actions of Z_p acting with the orbit genus $h > 0$. Let $e_+(p)$ and $e_0^{r,i}(p)$ be cardinalities of the above sets and observe that

$$e_0^r(p) \;\leq\; e_0^{r,1}(p) + e_0^{r,2}(p) + e_0^{r,3}(p) \quad \text{and} \quad e(p) \leq e_0^r(p) + e_+(p).$$

First, observe that by (31) we obtain a bound

$$e_+(p) \;\leq\; \sum_{h=1}^{[d/3]+1} e_h^{d-3(h-1)}(p) \;\leq\; ([d/3]+1)(p-1)^{d-1}.$$

Furthermore, notice that $e_k = e_{k,0} + e_{k,+}$, where $e_{k,0}$ is the cardinality of the set of all actions from E_k acting with the orbit genus 0, and $e_{k,+}$ is the number of remaining actions from E_k. The above gives the following.

$$e_{k,+} = \sum_{i=1}^{k} e_+(p_i) \leq k([d/3]+1)(p_k-1)^{d-1}.$$

Moreover,

$$\frac{e_{k,+}}{a_k} = ([d/3]+1)(d+3)!\frac{kp_k}{(p_k-1)^2(p_k-2)} \quad \xrightarrow{k\to\infty} \quad 0$$

Therefore, we can now focus on estimating $e_0^{r,i}(p)$.

First, consider the actions corresponding to the case (32.1). We have that $cbc^{-1} = b^\lambda$, for some integer $1 \leq \lambda < p$ for which $\lambda^q \equiv 1 \pmod p$. By Lemma 1.11 any epimorphism $\theta\colon \Delta \to Z_p$, defining an extendable action, maps $(x_1, \ldots, x_r)$ to

$$\big(\underbrace{b_1, b_1^\lambda, \ldots, b_1^{\lambda^{q-1}}}_{[b_1]}, \underbrace{b_2, b_2^\lambda, \ldots, b_2^{\lambda^{q-1}}}_{[b_2]}, \ldots, \underbrace{b_t, b_t^\lambda, \ldots, b_t^{\lambda^{q-1}}}_{[b_t]}\big), \tag{33}$$

for some nontrivial elements $b_i \in \langle b\rangle$. Suppose first that G is nonabelian. Then, marked segments in (33) are nontrivial orbits of the action of Z_q on Z_p, and a set R of representatives of all of them has cardinality $(p-1)/q$. Moreover, the product of all elements of any orbit is trivial, and thus, in particular, the topological type of the action depends only on the representatives of nontrivial orbits of the action of Z_q on Z_p, which can be chosen arbitrarily. Hence, for given q dividing r we have at most

$$((p-1)/q)^{r/q} = ((p-1)/q)^{(d+3)/q}$$

extendible action in the case (32.1) in this case. Now, assume that G is abelian. Then, we may assume that the vector (33) is of the form

$$\big(\underbrace{b_1, \ldots, b_t}, \overset{q}{\ldots}, \underbrace{b_1, \ldots, b_t}\big).$$

So, there are at most

$$(p-1)^{r/q} = (p-1)^{(d+3)/q}$$

of such vectors for a prime number q dividing r, in this case. Furthermore, we may assume that $b_1 = b$, by Theorem 1.3. So, to find an upper bound on the number of extendible actions in the case (32.1) we need to calculate the number of sequences $(b, b_2, \ldots, b_t)$ of nontrivial elements of Z_p which are representatives of all different nontrivial orbits of

action of Z_q on $Z_p - \{1\}$ in each of the two cases described above, and then sum the results. So, in particular

$$\begin{aligned} e_0^{r,1}(p) &\leq \sum_{q|r} ((p-1)/q))^{(d+3)/q-1} + \sum_{q|r} (p-1)^{(d+3)/q-1} \\ &\leq 2\sum_{q|r} (p-1)^{(d+3)/q-1} \\ &\leq 2(d+3)(p-1)^d, \end{aligned}$$

where the last inequality follows, since for $d \geq 1$, $(d+3)/q - 1 \leq d$ for all $q \geq 2$, while for $d = 0$, the only possibility is $q = 3$, and so $(d+3)/q - 1 = 0 \leq d$.

Thus,

$$e_{k,0}^1 = \sum_{j=1}^{k} e_0^{r,1}(p_i) \leq 2k(d+3)(p_k-1)^d \tag{34}$$

and finally by (30) and by (34)

$$\frac{e_{k,0}^1}{a_k} \leq 2(d+3)(d+3)! \frac{kp_k}{(p_k-1)(p_k-2)} \xrightarrow{k\to\infty} 0$$

which concludes the proof in this case.

Now consider the case (32.2). Then for $b = d^q$, $x_1, \ldots, x_r$ are mapped into

$$\tilde{a} = (\underbrace{b_1, \ldots, b_t}, \overset{q}{\ldots}, \underbrace{b_1, \ldots, b_t}, b),$$

where $b_i = b^{n_i}$ for some $n_i < p$, since we may fix the last elements of every generating vector by Theorem 1.3. Observe that such a vector is determined by a sequence $b_1, \ldots, b_t$ of elements of Z_p that satisfy some additional conditions. Therefore, there are at most $(p-1)^t$ extendable actions. However, if $d \geq 2$, then $t = (d+2)/q \leq d$ for all $q \geq 2$. Furthermore, if $d = 1$ then q must be equal to 3 and therefore this inequality is also true. Finally, if $d = 0$ then $q = 2$ and $t = 1$, so there is only one action of Z_p in this case by Theorem 1.3. So, the upper bound $(p-1)^d$ also holds in the case $d = 0$. All of these show that there are at most

$$\sum_{q|r-1} (p-1)^d \leq (d+2)(p-1)^d$$

classes of extendable actions. Therefore,

$$e_{k,0}^2 \leq k(d+2)(p_k-1)^d$$

which gives

$$\frac{e_{k,0}^2}{a_k} \leq (d+2)(d+3)! \frac{kp_k}{(p_k-1)(p_k-2)} \xrightarrow{k\to\infty} 0$$

and concludes the proof in this case. The case (32.3) is similar to (32.2), and we will not consider it in full detail. Notice that in this case, the generating vector for an action of Z_p is equal to

$$\tilde{a} = (\underbrace{b_1, \ldots, b_t}, \overset{q}{\ldots}, \underbrace{b_1, \ldots, b_t}, c_1, c_2),$$

which is determined by a sequence $b_1, \ldots, b_t, c_1$, since c_2 is determined by the proceeding elements by the definition of a generating vector. Hence, there are at most $(p-1)^{t+1}$ such actions. But, if $d \geq 3$, then $t+1 = (d+1)/q+1 \leq d$ for all $q \geq 2$. Furthermore, if $d = 2$, then q is forced to be equal to 3, so the upper bound on $t+1$ is also true in this case. This consideration gives that for $d > 1$

$$e_0^{r,3}(p) \leq (d+1)(p-1)^d. \tag{35}$$

Observe that this upper bound also holds in the case $d = 0$, since there are no actions of Z_p in this case that have the dimension 0. So, now we will consider the case $d = 1$, which is more complicated. Observe that if $d = 1$, then $r = 4$, and the only possibility is $q = 2$. So, in this case, the generating vector for an action of Z_p is equivalent to

$$(b_1, b_1, c_1, c_2).$$

However, such a generating vector is equivalent to

$$(b, b, c_1', c_2'),$$

where b is a fixed generator of Z_p, by Theorem 1.3. Therefore, there are at most $(p-2)$ such actions, since c_2' is uniquely determined by c_1'. So, inequality (35) is true in this case. Therefore, we get that

$$\frac{e_{k,0}^3}{a_k} \leq (d+1)(d+3)! \frac{kp_k}{(p_k-1)(p_k-2)} \xrightarrow{k\to\infty} 0.$$

And finally, since $e_k \leq e_{k,0}^1 + e_{k,0}^2 + e_{k,0}^3 + e_{k,+}$ and $m_k = a_k - e_k$,

$$\frac{m_k}{a_k} = \frac{a_k - e_k}{a_k} = 1 - \frac{e_k}{a_k} \xrightarrow{k\to\infty} 1,$$

which concludes the proof. □

5. On genera corresponding to connected singular loci

We start this section by introducing some notation that will be used throughout the rest of this thesis. So, let p_n denote the n-th prime number and $q_n = (p_n - 1)/2$. For a prime number $p_n = p > 3$ by $\mathrm{lcm}(p)$ we mean the number $\mathrm{lcm}(q_3, q_4, \ldots, q_n)$. Suppose that there is a solution to the following system of congruences:

$$\begin{cases} x \equiv -1 \pmod{q_i} & \text{for every } 3 \leq i \leq n \\ x \equiv h \pmod{q_{n+1}} & \text{for some } 0 \leq h \leq q_{n+1} - 2, \end{cases} \tag{36}$$

and let g be the smallest positive integer satisfying this system. By $\mathrm{Ar}^h_{p_{n+1}}$, we denote the arithmetic sequence of all positive solutions to the system (36), i.e., the sequence

$$g + nr,$$

where $r = \mathrm{lcm}(p_{n+1})$. In addition, by Ar^0_5 we understand the sequence of positive even numbers. Let g be an element of $\mathrm{Ar}^h_{p_{n+1}}$. Recall that by Lemma 1.1, the orbit genus of an action of Z_{p_k}, for any $k \leq n$, on a compact Riemann surface of genus g, has the remainder $q_k - 1$ while dividing by q_k. Thus, Lemma 1.7 implies that there are no finitely maximal actions of Z_{p_k} on compact Riemann surfaces of genus g. However, there may exist finitely maximal actions of $Z_{p_{n+1}}$. For this reason, the above sequences will be particularly useful for our purposes, since a construction of maximal Z_p-actions is the main tool used to determine whether $\mathcal{S}_g$ is connected. The following is a consequence of Theorem 4.2, i.e. the main result in [32].

Lemma 5.1. *Let $p \geq 5$ be a prime, $0 \leq h \leq (p-5)/2$ be an integer and g be an integer such that $(p-1)/2$ divides $g-h$ and $g \geq p^2-3$. Then, the singular locus $\mathcal{S}_g$ is disconnected.*

Proof. Let

$$r = \frac{2(g-1) - 2p(h-1)}{p-1}.$$

Observe that by our assertion, r is an integer. Moreover, since $h \leq (p-5)/2$ and $g \geq p^2-3$ we get that $r \geq p + 8$. By Theorem 4.2 and the Riemann-Hurwitz formula, there exists a finitely maximal action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$ on a compact Riemann surface of genus g. This concludes the proof by Lemma 1.21. □

Now, observe that if g is an element of a sequence $\mathrm{Ar}^h_{p_{n+1}}$ such that $g \geq 4p_n^2 - 3$, then $\mathcal{S}_g$ is disconnected. To see this, notice that by Bertrand's Postulate

$$g \geq 4p_n^2 - 3 = (2p_n)^2 - 3 > p_{n+1}^2 - 3, \tag{37}$$

and the assertion follows from Lemma 5.1. Furthermore, the smallest element of the sequence $\mathrm{Ar}^h_{p_{n+1}}$ is bounded below by $\mathrm{lcm}(p_n)-1$, since it satisfies the first $n-2$ congruences of (36). Therefore, we now want to focus on finding a suitable lower bound on $\mathrm{lcm}(p)$ that will serve our purposes.

5.1. **Some analytic number theory.** We start this subsection with the well-known Linnik's Theorem [29].

Theorem 5.2. *There exist positive constants c and L such that for every co-prime positive integers a and r such that $a < r$, the smallest prime number in the sequence $a + nr$ is smaller than cr^L.* □

Corollary 5.3. *Let $p \geq 5$ be a prime number. Then,* $\operatorname{lcm}(p)$ *is divisible by all primes $q \leq (p/c)^{1/L}$.*

Proof. First, observe that this number is divisible by 2. Let q be an odd prime smaller than or equal to $(p/c)^{1/L}$, consider a sequence $1+nq$. By Theorem 5.2, the smallest prime in this progression is smaller than $cq^L \leq p$. Thus, there is a prime $r \leq p$ such that $r-1$ is divisible by q, since q is odd the number $(r-1)/2$ is also divisible by q. This concludes the proof. □

The proof of the main result of this subsection is based on the asymptotic properties of the First Chebyshev and the primorial functions, which are defined as

$$\vartheta(x) = \sum_{p \leq x} \ln p \quad \text{and} \quad x\# = \prod_{p \leq x} p$$

respectively. Their asymptotic behavior is well studied and the following

Theorem 5.4.

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$$

□

which is equivalent to the Prime number Theorem, and whose proof can be found in [3], is perhaps the most relevant one. However, we will use a different result, which is more suitable for our purposes.

Lemma 5.5. *$x\# > 2^x$, for $x \geq 563$.*

Proof. By Theorem 4 in [34] for every $x \geq 563$

$$\vartheta(x) > x\left(1 - \frac{1}{2\ln x}\right).$$

To conclude the proof, we need to show that $1 - \frac{1}{2\ln x} > \ln 2$, for $x \geq 563$. But this is the case for $x > e^2$, since $1 - \frac{1}{2\ln x}$ is an increasing function.

□

The following is the direct result of Lemma 5.5 and Corollary 5.3.

Proposition 5.6. $\operatorname{lcm}(p) > 2^{(p/c)^{1/L}}$ *for p large enough.* □

So, we get the following lemma, which is what we seek.

Lemma 5.7. $\operatorname{lcm}(p) \geq 4p^2 - 2$, *for p large enough.* □

5.2. **Number of genera corresponding to connected singular loci.** This subsection is devoted to finding genera corresponding to connected singular loci. This topic appeared to be closed since the paper [6], in which the authors showed that the singular locus is connected exactly for genera

$$3, 4, 7, 13, 17, 19, 59. \tag{38}$$

However, the proof of this statement is flawed. Furthermore, the assertion itself is not correct, and we begin by pointing out the gap, which occurs in the proof of Theorem 3.1 in [6]. In the third case the authors construct actions of Z_7 with the signature $(2; 7, \overset{r}{\ldots}, 7)$ and claim that these actions are finitely maximal. However, an action of Z_7 with the orbit genus 2 is never finitely maximal by Theorem 5 in [32].

This can also be easily seen using Lemma 1.14, since for every generating vector of Z_p, the length of a q-tail equals at most $p-1$. Therefore, the length of the 2-tail of any $(2; 7, \overset{r}{\ldots}, 7)$-generating vector for an action of Z_7 equals at most $6 = 2 \cdot 2 + 2$, and the assertion follows. Recall that the existence of the exceptional component was established in [8]; here we present a simpler proof of this fact.

Lemma 5.8. *All strata corresponding to actions of Z_2 and Z_3 are contained in the same connected component $\mathcal{C}_g^{2,3}$.*

Proof. Observe that the locus composed of points corresponding to actions of Z_2 is connected by Theorem 3.14 in [23]. Moreover, by Lemma 1.14, every Cornalba component corresponding to an action of Z_3 intersects nontrivially with this locus, which concludes the proof. □

The fact that all Cornalba components corresponding to actions of Z_2 on compact Riemann surfaces of genus g are contained in the same connected component is the crucial property used to determine whether $\mathcal{S}_g$ is connected. Based on this, we have the following.

Theorem 5.9. *The singular locus is connected for $g = 149$.*

Proof. Observe that by Lemma 5.8 we only need to prove that every action of a group of prime order on a compact Riemann surface of genus 149 extends to an action of a group of even order.

First, observe that $2g+1$ is not a prime number and hence $\mathcal{S}_{149}$ contains no isolated points, by Lemma 1.26. Moreover, 149 is itself a prime number, so the only p-gonal actions are those of orders 2 and 3 by Lemma 1.1. Observe, that $g \equiv -1 \pmod{(p-1)/2}$ for primes $p = 5, 7, 11, 13$, and so the orbit genus of any action of Z_p is at least $(p-3)/2$, by Lemma 1.1, which by Lemma 1.7 is sufficient for our purposes.

Suppose, to get a contradiction, that there is an action of Z_p with the signature $(h; p, \overset{r}{\ldots}, p)$ on a compact Riemann surface of genus 149 that does not extend to an action of a group

of even order. Then, $r \geq 2h+3$ by Lemma 1.14 and $p \geq 17$. By the Riemann-Hurwitz formula

$$2(g-1) = 2p(h-1) + r(p-1) \geq 2p(h-1) + (2h+3)(p-1) \geq 34(h-1) + 16(2h+3).$$

This easily gives $h \leq 4$. We will consider every case separately. Recall that $h > 0$.

Let $h = 1$. Then, the only prime number $p \geq 17$ such that $g-1$ is divisible by $(p-1)/2$ is $p = 149$, but by the Riemann-Hurwitz formula, an action of Z_{149} has only two fixed points, which concludes the proof in this case by Lemma 1.7.

For $h = 2$, the only possibility is $p = 43$. Using the Riemann-Hurwitz formula, we can calculate that the number of fixed points of an action of Z_{43} with the orbit genus 2 equals 5. Thus, every such action extends to an action of a group of even order, by Lemma 1.14.

For $h = 3$, the only $p \geq 17$ that satisfies the condition of Lemma 1.1 is $p = 293$. However, by the Riemann-Hurwitz formula, there is no action of Z_{293} on a compact Riemann surface of genus 149.

For $h = 4$, the prime we need to consider is $p = 59$, but there is no action of Z_{59} on a surface of genus 149 by the Riemann-Hurwitz formula. This concludes the proof. □

Remarks 5.10. (1) Using the same strategy as above, namely the one based on Lemma 1.14, it can be verified through a straightforward calculation that $\mathcal{S}_g$ for g given in the list (38) are connected.

The above Proposition shows that the problem of finding the genera corresponding to connected singular loci is not yet closed. For now, it is not even clear whether there are finitely many such genera. Our goal is to prove that this is, in fact, the case.

Theorem 5.11. *The singular loci are connected for only a finite number of genera.*

Proof. Let $g \geq 2$ be an integer. Observe that g is either even or satisfies the system (36) for some prime number $p_{n+1} = p$ and $h \leq (p-5)/2$. Thus, it is an element of a sequence Ar_p^h. Recall that the smallest element in this sequence is greater than or equal to $\mathrm{lcm}(p_n) - 1$. Therefore, if p is large enough, then every element of Ar_p^h is greater than $4p_n^2 - 3$, by Lemma 5.7. Consequently, $g \geq p_{n+1}^2 - 3$ by (37), which means that $\mathcal{S}_g$ is disconnected by Lemma 5.1. Hence, there are only finitely many sequences Ar_p^h that may contain a number corresponding to a connected singular locus. Recall that, by Lemma 5.1, sufficiently large elements of Ar_p^h correspond to disconnected singular loci. Thus, the result follows. □

Remarks 5.12. The proof of the above theorem relies on the inequality from Lemma 5.7. Available evidences suggest that it holds for all primes greater than 19. If this is indeed the case, than it can be proved that singular loci are connected only for genera $3, 4, 7, 13, 17, 19, 59$, and 149. We conjecture that this is, in fact, the case.

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