

UNIVERSITY OF GDAŃSK

Certified Domination in Graphs Structural Properties and Applications

by

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Abstract

This doctoral dissertation focuses on the concept of certified dominating sets in graphs, introduced by Jerzy Topp et al. [26], in which each member vertex has either zero or at least two neighbors outside the set. This condition prevents the sets from being “uncertified” which could otherwise make them more vulnerable. The study employs a complex methodology, combining rigorous theoretical analysis, algorithm design, and practical applications. The majority of the work is devoted to the fundamental properties of the certified domination number, $\gamma_{cer}(G)$, and its relationship to the classical domination number, $\gamma(G)$, particularly in the case of $\mathcal{DD}_2$ -graphs, which can be split into a dominating set and a 2-dominating set. The study provides structural characterizations of important graph classes, such as vertex-certified-domination-critical (γ_{cer} -critical) graphs, in which removing any vertex reduces the certified domination number, and minimal $\mathcal{DD}_2$ -graphs. The latter concept, originally proposed by Jerzy Topp [84], has subsequently been extended to other variants of domination-type structures [56, 57]. A key result is a novel theorem stating that $\gamma(G) = \gamma_{cer}(G)$ if and only if the graph has a minimum dominating set D such that $V_G - D$ forms a 2-dominating set. The dissertation further establishes that deciding the inequality $\gamma(G) \neq \gamma_{cer}(G)$ is NP-hard, even for graphs with only one weak support vertex. To address this problem, a new linear-time algorithm for finding the smallest certified dominating set in trees is developed, along with a corrected and validated Integer Linear Programming (ILP) formulation for general graphs. Two new heuristic algorithms, “ClassicToCert” and “ApproxCert”, are also proposed and systematically evaluated. Experimental results demonstrate that the constructive approach yields superior solutions compared to approaches that modify an existing dominating set. These theoretical and algorithmic contributions are motivated by practical applications, such as improving the efficiency of fire safety water networks and analyzing the structure of social networks. The dissertation concludes by presenting a new set of tools for analyzing graph domination and suggesting several promising directions for future research, including the study of k -certified domination and the development of parameterized algorithms.

Streszczenie

Niniejsza rozprawa doktorska dotyczy zbiorów dominujących z certyfikatami w grafach, zdefiniowanych przez Jerzego Toppa i in. [26], w których każdy należący do nich wierzchołek ma zero lub co najmniej dwóch sąsiadów poza tym zbiorem. Warunek ten zapobiega sytuacji, w której zbiory te stają się “pozbawione certyfikatów”, co w przeciwnym razie mogłoby uczynić je bardziej podatnymi na zagrożenia. W badaniach zastosowano złożoną metodologię, łączącą rygorystyczną analizę teoretyczną, projektowanie algorytmów i praktyczne zastosowania. Większość pracy poświęcono podstawowym własnościom liczby dominowania z certyfikatami, $\gamma_{cer}(G)$, oraz jej związkowi z klasyczną liczbą dominowania, $\gamma(G)$, w szczególności w przypadku grafów $\mathcal{DD}_2$, które można podzielić na zbiór dominujący i zbiór 2-dominujący. Praca dostarcza strukturalnych charakterystyk ważnych klas grafów, takich jak grafy krytyczne ze względu na wierzchołkowy zbiór dominujący z certyfikatami (grafy γ_{cer} -krytyczne), w których usunięcie dowolnego wierzchołka zmniejsza liczbę dominowania z certyfikatami, oraz minimalne grafy $\mathcal{DD}_2$. Pojęcie to, pierwotnie zaproponowane przez Jerzego Toppa [84], zostało następnie rozszerzone na inne strukturalne warianty [56, 57]. Kluczowym wynikiem jest nowe twierdzenie mówiące, że $\gamma(G) = \gamma_{cer}(G)$ wtedy i tylko wtedy, gdy graf posiada najmniejszy zbiór dominujący D taki, że $V_G - D$ tworzy zbiór 2-dominujący. Rozprawa dowodzi ponadto, że rozstrzygnięcie, czy zachodzi nierówność $\gamma(G) \neq \gamma_{cer}(G)$ jest problemem NP-trudnym, nawet dla grafów posiadających tylko jeden słaby wierzchołek wspierający. Aby rozwiązać ten problem, opracowano nowy algorytm działający w czasie liniowym do znajdowania najmniejszego zbioru dominującego z certyfikatami w drzewach, wraz ze skorygowanym i zweryfikowanym sformułowaniem problemu w postaci programowania liniowego całkowitoliczbowego (ILP) dla grafów ogólnych. Zaproponowano również i systematycznie oceniono dwa nowe algorytmy heurystyczne: “ClassicToCert” oraz “ApproxCert”. Wyniki eksperymentalne pokazują, że podejście konstruktywne daje lepsze rozwiązania w porównaniu do podejść modyfikujących istniejący zbiór dominujący. Te teoretyczne i algorytmiczne wkłady są motywowane praktycznymi zastosowaniami, takimi jak poprawa wydajności przeciwpożarowych sieci wodociągowych oraz analiza struktury sieci społecznych. Rozprawę kończy przedstawienie nowego zestawu narzędzi do analizy dominowania w grafach oraz sugestie dotyczące kilku obiecujących kierunków przyszłych badań, w tym badania zbiorów k -dominujących z certyfikatami i rozwoju algorytmów sparametryzowanych.

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List of Publications Included in the Dissertation

1. **Paper A** [25]: M. Dettlaff, M. Lemańska, M. Miotk, J. Topp, R. Ziemann, P. Żyliński, *Graphs with equal domination and certified domination numbers*, *Opuscula Mathematica* 39(6) (2019), 815–827.
2. **Paper B** [80]: M. Miotk, *Certified domination critical graphs upon vertex removal*, *Utilitas Mathematica* 123 (2025), 179–194.
3. **Paper C** [81]: M. Miotk, *On a class of graphs with equal domination and certified domination numbers*, *Discrete Mathematics Letters* 16 (2025), 29–34.
4. **Paper D** [84]: M. Miotk, J. Topp, P. Żyliński, *Disjoint dominating and 2-dominating sets in graphs*, *Discrete Optimization* 35 (2020), 100553.
5. **Paper E** [95]: J. Raczek, M. Miotk, *Theoretical modelling of efficient fire safety water networks by certified domination*, *Scientific Reports* 14 (2024), 21574.
6. **Paper F** [96]: J. Raczek, M. Miotk, *Two approaches to constructing certified dominating sets in social networks*, *IEEE Access* 13 (2025), 17495–17505.

Other Publications by the Author

1. **Paper G** [82]: M. Miotk, J. Topp, *Hangable graphs*, International Journal of Advances in Mathematics 2018(4) (**2018**), 44–50.

We define hangable graphs as graphs in which the periphery of every vertex is contained in the periphery of the graph, and we establish characterizations of this class for selected graph operations.

2. **Paper H** [77]: A. Lingas, M. Miotk, J. Topp, P. Żyliński, *Graphs with equal domination and covering numbers*, Journal of Combinatorial Optimization 39 (**2019**), 55–71.

We provide equivalent characterizations of connected graphs for which the domination number equals the covering number, as well as of bipartite graphs in which the domination number is equal to the cardinality of the smaller partite set. We also present a constructive characterization of trees belonging to this class and describe a polynomial-time recognition algorithm.

3. **Paper I** [83]: M. Miotk, J. Topp, P. Żyliński, *Bipartization of graphs*, Graphs and Combinatorics 35(5) (**2019**), 1169–1177.

We introduce the notion of the bipartization of a graph with respect to a function and apply this operation to characterize bipartite graphs in which the domination number equals the cardinality of the smaller partite set.

4. **Paper J** [86]: M. Miotk, P. Żyliński, *Spanning Trees with Disjoint Dominating and 2-Dominating Sets*, Discussiones Mathematicae Graph Theory 42(1) (**2022**), 299–308.

We characterize graphs that admit a spanning tree with disjoint dominating and 2-dominating sets by means of structural conditions involving weak support vertices and their neighborhoods.

5. **Paper K** [85]: M. Miotk, M. Zakrzewski, P. Żyliński, *Unique paired vs edge-vertex minimum dominating sets in trees*, Utilitas Mathematica 125 (**2025**), 103–108.

We prove that a tree has a unique minimum edge-vertex dominating set if and only if it has a unique minimum paired dominating set. Moreover, we show that all such minimum sets are supported on the same vertex set.

Chapter 1

Introduction

In graph theory, domination is a central concept that involves selecting a subset of vertices such that every vertex in the graph either belongs to this subset or is adjacent to at least one vertex in it. This concept models scenarios where a relatively small set of vertices can monitor or control an entire network, such as in social, biological, or communication systems. Domination has a rich and well-documented history. For example, the defensive strategies of the Roman Empire can be interpreted through the lens of domination [19]. In the 19th century, related questions arose in recreational mathematics, particularly in the classic problem of placing queens on a chessboard so that every square is under attack. This problem was investigated by, among others, Max Bezzel and Carl Jaenisch [7, 24].

The formal mathematical study of domination began in the early 20th century. In 1936, Hungarian mathematician Dénes König introduced the concept of a *basis of the second kind*, defined as a set of control points that are not directly linked to each other [74]. In 1944, John von Neumann and Oskar Morgenstern independently identified a similar structure in the context of game theory, referring to it as a *kernel* [113]. The term *domination*, in its modern sense, was first used by Øystein Ore in 1962. In the same year, Claude Berge highlighted the importance of the minimum number of vertices required to control a network [5]. Ore also formulated a basic principle: to dominate an entire network with no isolated vertices, it suffices to select a carefully chosen subset of its vertices [91].

The field developed rapidly, driven by the work of mathematicians such as Tibor Gallai [31] and Vadim Vizing, who posed a well-known question about how domination behaves when networks are combined [112]. In the 1970s, Ernest J. Cockayne and Stephen T. Hedetniemi published influential papers that helped standardize the terminology and introduced numerous new ideas [18, 21]. The two books written by Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater in 1998 became standard references in the field [45, 46]. Their later editions, co-authored with Michael A. Henning, remain significant today [43, 44].

This dissertation contributes to the field by introducing and analyzing two novel concepts. The first concept, *certified domination*, originally proposed by Professor Jerzy Topp, addresses the need for networks that can be independently verified and monitored. A certified dominating set is a collection of vertices that not only dominates the entire network but also ensures that each vertex in the set is *certified* by its neighbors outside the set. In other words, a vertex in the dominating set must either have no neighbors outside the set or be adjacent to at least two vertices outside the set. The second concept concerns a particular type of network called *DD₂-graphs*. A network belongs to this class if its vertices can be partitioned into two sets, one forming a dominating set and the other providing additional 2-domination, such that every vertex in the second set is adjacent to at least two vertices outside it. This concept was introduced by Michael A. Henning and Douglas F. Rall [54]. The appended papers examine the theoretical properties and structural implications of these two concepts, thereby extending classical domination theory.

1.1 Graph Theoretic Concepts

A graph G is defined as an ordered pair $(V(G), E(G))$, where *vertices* $V(G)$ is the set of vertices and *edges* $E(G)$ is the set of edges, each being an unordered pair of distinct vertices. The *order* of G is denoted $n(G) = |V(G)|$, and its *size* is denoted $m(G) = |E(G)|$. The open neighborhood of a vertex $v \in V(G)$ is defined as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The closed neighborhood is denoted $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v , denoted $\deg_G(v)$, is the number of edges incident to it, which equals the cardinality of its open neighborhood, $N_G(v)$. The symbols $\delta(G)$ and $\Delta(G)$ represent the minimum and maximum degrees of G , respectively.

A graph is *regular* if all its vertices have the same degree. A vertex of degree 1 is called a *leaf*, and its adjacent vertex is called a *support vertex*. A support vertex is classified as *strong* if it is adjacent to at least two leaves, and as *weak* if it is adjacent to only one leaf. A vertex with degree 0 is isolated, while a *universal* vertex is one that is adjacent to all other vertices.

The *complement* of a graph G , denoted $\overline{G}$, has vertex set $V(G)$ and edge set $\{uv \mid u, v \in V(G), u \neq v, uv \notin E(G)\}$. A *graph homomorphism* from a graph G to a graph H is a function $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph* of G . The *subgraph induced by S* , denoted $G[S]$, has vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$.

A *path* is a trail in which all vertices (and therefore all edges) are distinct; that is, it is a sequence of adjacent vertices with no repetition. The distance $d_G(u, v)$ is the length of the shortest path between vertices u and v . The *diameter* of a graph is the maximum distance between any two vertices. A graph is *connected* if there exists a path between every pair of vertices.

A *path graph* on $k + 1$ vertices, denoted P_{k+1} , has vertex set $V(P_{k+1}) = \{v_0, \dots, v_k\}$ and edge set $E(P_{k+1}) = \{v_{i-1}v_i \mid i = 1, \dots, k\}$. A *cycle graph* on k vertices (for $k \geq 3$), denoted C_k , has vertex set $V(C_k) = \{v_0, \dots, v_{k-1}\}$, and its edge set is obtained by taking the edges of the path P_k and adding an edge between its endpoints, giving $E(C_k) = \{v_{i-1}v_i \mid i = 1, \dots, k-1\} \cup \{v_{k-1}v_0\}$. The *wheel graph* W_n (for $n \geq 4$) is formed by connecting a single universal vertex to all vertices of a cycle C_{n-1} . A *complete graph*, denoted K_n , is a graph in which every pair of distinct vertices is joined by an edge.

An *acyclic graph* is a graph that contains no cycles. A *tree* is a connected acyclic graph. A special type of tree is a *star graph*, $K_{1,n-1}$, which consists of one central vertex adjacent to $n - 1$ leaves. A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets, U and V , such that every edge connects a vertex in U to a vertex in V .

A graph is *planar* if it can be drawn in the plane without any edges crossing. A graph is *P_4 -free* if it does not contain an induced path on four vertices, and *claw-free* if it does not contain an induced $K_{1,3}$. A *chordal graph* is a graph in which every cycle of four or more vertices has a chord (i.e., an edge not on the cycle that connects two nonconsecutive vertices), and an *interval graph* is a graph whose vertices can be represented as intervals on the real line, with an edge between two vertices if and only if their intervals intersect.

The *Cartesian product* $G \square H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. A *subdivision* of a graph is formed by replacing edges with paths of length one or more. Adding a new leaf v' and the edge vv' for each vertex $v \in V(G)$ produces the *corona* $G \circ K_1$.

A *vertex cover* is a subset of vertices such that every edge in the graph has at least one endpoint in the set; its smallest possible size is called the *vertex cover number*, $\beta(G)$. An *independent set* is a subset $S \subseteq V(G)$ in which no two vertices are adjacent. A set of edges with no shared endpoints is called a *matching*, and a *perfect matching* is a matching that covers every vertex in the graph.

A subset $S \subseteq V(G)$ is a *dominating set* if every vertex $v \in V(G) - S$ is adjacent to at least one vertex in S . The *domination number*, denoted $\gamma(G)$, is the minimum size of a dominating set, and a set S with $|S| = \gamma(G)$ is referred to as a $\gamma(G)$ -set. A dominating

set is *minimal* if no proper subset of it is also a dominating set. Conversely, the *upper domination number*, $\Gamma(G)$, is the maximum size of a minimal dominating set. The *private neighborhood* of a vertex $v \in S$ with respect to S is defined as $pn(v, S) = N[v] - N[S - \{v\}]$. Any vertex in $pn(v, S)$ is called a *private neighbor* of v .

1.2 Domination Variations

In graph theory, domination is a fundamental concept, and many related variants of domination have been introduced to capture specific structural properties or model particular applications [44, 46]. Generally, these variants impose additional conditions either on the dominating set itself or on its interaction with the rest of the graph. In particular, some variants restrict the internal structure of the dominating set S . Below, we introduce those variants of domination that are relevant to the results presented in this dissertation.

A set S is an *independent dominating set* if it is both a dominating set and an independent set. The smallest size of such a set is called the *independent domination number*, denoted $i(G)$ [5]. If every vertex in the graph, including those in S , is adjacent to at least one vertex in S , then S is a *total dominating set*, and the minimum size of such a set is the *total domination number*, denoted $\gamma_t(G)$ [18]. A set S is a *paired dominating set* if it is a dominating set and the subgraph induced by S contains a perfect matching. The minimum size of a paired dominating set is the *paired domination number*, denoted $\gamma_{pr}(G)$ [49]. A set S is a *connected dominating set* if it is a dominating set and the subgraph induced by S is connected; its minimum size is the *connected domination number*, denoted $\gamma_c(G)$ [104].

A set S is a *2-dominating set* if every vertex $v \in V(G) - S$ is adjacent to at least two vertices in S ; the smallest size of such a set is the *2-domination number*, denoted $\gamma_2(G)$ [28]. More generally, a set S is a *k-tuple dominating set* if every vertex in $V(G) - S$ is adjacent to at least k vertices in S ; the minimum size of such a set is the *k-tuple domination number*, denoted $\gamma_{\times k}(G)$ [76]. A set S is a *k-hop dominating set* if every vertex in $V(G) - S$ is within distance k of some vertex in S ; the minimum size of such a set is the *hop domination number*, denoted $\gamma_k(G)$ [79]. In vertex-edge domination, a vertex $v \in V(G)$ dominates every edge incident to v as well as every edge adjacent to those edges. A set $S \subseteq V(G)$ is a vertex-edge dominating set if every edge of G is dominated by at least one vertex in S [93].

Other variations of domination arise from considerations of functionality and efficiency. A dominating set S is a *perfect dominating set* if every vertex in $V(G) - S$ is adjacent to exactly one vertex in S ; the minimum size of such a set is the *perfect domination number*, denoted $\gamma_p(G)$ [9]. A set S is an *efficient dominating set*, also known as a *perfect code* in coding theory, if every vertex in the graph is dominated by exactly one vertex from S ; the minimum size of such a set is the *efficient domination number*, denoted $\gamma_e(G)$ [4, 50]. A dominating set is *accurate* if for any distinct vertices $u, v \in S$, $pn(u, S) \neq pn(v, S)$; the minimum size of such a set is the *accurate domination number*, denoted $\gamma_a(G)$ [72].

Roman domination is defined by a function $f: V(G) \rightarrow \{0, 1, 2\}$ such that any vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The objective is to minimize the total weight of the function, denoted $\gamma_R(G)$ [19]. Similarly, *k-rainbow domination* is defined by a function $f: V(G) \rightarrow 2^{\{1, \dots, k\}}$ such that for any vertex v with $f(v) = \emptyset$, the union of the sets assigned to its neighbors satisfies $\bigcup_{u \in N(v)} f(u) = \{1, \dots, k\}$; the minimum weight, $\sum_{v \in V(G)} |f(v)|$, is called the *k-rainbow domination number*, denoted $\gamma_{rk}(G)$ [12]. The *inverse domination number*, denoted $\gamma^{-1}(G)$, is the minimum size of a dominating set D such that $V(G) - D$ contains another dominating set [71].

Chapter 2

Certified Domination: Theory, Structure, and Applications

A set $S \subseteq V(G)$ is called a *power dominating set* if, starting from the closed neighborhood $N[S]$, all vertices of G can be observed by repeatedly applying the rule that any observed vertex with exactly one unobserved neighbor forces that neighbor to become observed. The smallest size of such a set is the *power domination number* of G , denoted $\gamma_P(G)$. This concept was introduced by Teresa W. Haynes, Sandra M. Hedetniemi, Stephen T. Hedetniemi, and Michael A. Henning [42].

A closely related, innovative variant is *certified domination*, introduced by Jerzy Topp et al. in [26]. A dominating set $D \subseteq V(G)$ is *certified* if every vertex in D has either zero or at least two neighbors in $V(G) \setminus D$. The smallest size of a certified dominating set of G is denoted $\gamma_{\text{cer}}(G)$. Below, we briefly describe selected results on certified domination obtained in [26].

It is evident that every support vertex must be included in any certified dominating set. There are several graph families for which the exact values of $\gamma_{\text{cer}}(G)$ are known. For a path with n vertices, $\gamma_{\text{cer}}(P_n) = \lceil n/3 \rceil$ when $n = 3$ or $n \geq 5$, with the remaining small cases given by $\gamma_{\text{cer}}(P_1) = 1$, $\gamma_{\text{cer}}(P_2) = 2$, and $\gamma_{\text{cer}}(P_4) = 4$. For cycles with $n \geq 3$, $\gamma_{\text{cer}}(C_n) = \lceil n/3 \rceil$.

For complete graphs K_n , the certified domination number is $\gamma_{\text{cer}}(K_n) = 1$ if $n = 1$ or $n \geq 3$, and $\gamma_{\text{cer}}(K_n) = 2$ if $n = 2$. For complete bipartite graphs $K_{m,n}$ with $1 \leq m \leq n$, $\gamma_{\text{cer}}(K_{m,n}) = 1$ when $m = 1$ and $n > 1$ (i.e., a star graph), and $\gamma_{\text{cer}}(K_{m,n}) = 2$ otherwise (i.e., for $m \geq 2$). In the case of wheel graphs W_n , $\gamma_{\text{cer}}(W_n) = 1$ for $n \geq 4$. Furthermore, for any graph G with at least three vertices, $\gamma_{\text{cer}}(G) = 1$ if and only if G contains a universal vertex.

The certified domination number is closely related to various graph parameters. Dettlaff et al. [26] established an upper bound, showing that for any graph G , the inequality

$\gamma_{\text{cer}}(G) \leq \gamma(G) + |S_1(G)|$ holds, where $S_1(G)$ denotes the set of weak support vertices [26, Theorem 3.3]. This implies the simpler bound $\gamma_{\text{cer}}(G) \leq 2\gamma(G)$ for corona graphs.

Conditions for equality, stated as $\gamma_{\text{cer}}(G) = \gamma(G)$, have also been characterized. For a connected graph G with $n \geq 3$, equality holds if and only if there exists a minimum dominating set that does not contain all non-leaf neighbors of any weak support vertex [26, Lemma 4.2]. Consequently, equality also holds if G has no weak support vertices or if G has exactly one minimum dominating set [26, Corollaries 4.1, 4.4].

Extreme cases have been examined as well. Specifically, if every non-trivial component of G is either the complement of a complete graph or a corona graph, then $\gamma_{\text{cer}}(G) = n$ [26, Theorem 5.3]. Additionally, $\gamma_{\text{cer}}(G)$ cannot equal $n - 1$ for $n \geq 3$. The behavior of $\gamma_{\text{cer}}(G)$ under various graph operations has also been studied. For example, adding a non-leaf vertex can increase the certified domination number by at most 1 [26, Theorem 6.3].

Finally, for any graph G of order $n \geq 5$, Nordhaus–Gaddum-type results establish tight bounds on the certified domination number: $\gamma_{\text{cer}}(G) + \gamma_{\text{cer}}(\overline{G}) \leq n + 2$ and $\gamma_{\text{cer}}(G)\gamma_{\text{cer}}(\overline{G}) \leq 2n$ [26, Theorem 7.6].

By adding further graph-theoretic constraints to the definition of a certified dominating set, researchers have developed several specialized variants. One notable extension is *certified perfect domination*. A set $S \subseteq V(G)$ is certified perfect dominating if it is certified and each vertex $u \notin S$ has exactly one neighbor in S . The minimum size of such a set is referred to as the *certified perfect domination number*, $\gamma_{\text{cerp}}(G)$. For paths P_n and cycles C_n , this number equals $\gamma_{\text{cerp}}(G) = n/3$ when $n \equiv 0 \pmod{3}$ [38].

Another variant, *hop certified domination*, combines the conditions of hop domination and certified domination. The minimum size of such a set is the *hop certified domination number*, $\gamma_{\text{cerh}}(G)$. By definition, $\gamma_{\text{cerh}}(G) \geq \gamma_h(G)$, but the parameters $\gamma_{\text{cerh}}(G)$ and $\gamma_{\text{cer}}(G)$ are not generally comparable. For instance, in K_n with $n \geq 3$ one has $\gamma_{\text{cer}}(K_n) = 1$ while $\gamma_h(K_n) = \gamma_{\text{cerh}}(K_n) = n$, whereas for a path P_n , $\gamma_{\text{cerh}}(P_n) = n$, and $\gamma_{\text{cer}}(P_n) = 1$ for $n = 1$, $\gamma_{\text{cer}}(P_n) = 2$ for $n = 2$, $\gamma_{\text{cer}}(P_n) = 4$ for $n = 4$, and $\gamma_{\text{cer}}(P_n) = \lceil n/3 \rceil$ for $n = 3$ or $n \geq 5$ [41].

A further extension, *accurate certified domination*, combines the properties of accurate domination and certified domination. A set is an accurate certified dominating set if it is both certified and accurate. The minimum size of such a set is the *accurate certified domination number*, $\gamma_{\text{acer}}(G)$. This parameter satisfies the inequalities $\gamma(G) \leq \gamma_a(G) \leq \gamma_{\text{acer}}(G)$ and $\gamma_{\text{cer}}(G) \leq \gamma_{\text{acer}}(G)$. For any graph H , the accurate certified domination number of its corona with K_1 is $\gamma_{\text{acer}}(H \circ K_1) = |V(H)|$ [8].

Finally, *connected certified domination* requires a certified dominating set to induce a connected subgraph. The minimum size of such a set is the *connected certified domination number*, $\gamma_{\text{ccer}}(G)$. For a connected graph G of order $n \geq 3$, $\gamma_{\text{ccer}}(G) = 1$ if and only if G contains a universal vertex. For paths and cycles of order $n \geq 4$, $\gamma_{\text{ccer}}(P_n) = \gamma_{\text{ccer}}(C_n) = n$ [62, 78].

2.1 Properties of Certified Domination (Paper A [25])

A central topic in domination theory is the comparison of various domination parameters and the characterization of graphs for which two such parameters are equal. For example, Haynes, Hedetniemi, and Henning demonstrated that for a tree T , the equality $\gamma(T) = i(T)$ holds if and only if T is a corona graph $H \circ K_1$ [44]. They further showed that for a tree T of order $n \geq 2$, the equality $\gamma(T) = \gamma_t(T)$ holds if and only if T is a disjoint union of paths P_2 or P_4 . Additionally, it is known that $\gamma(G) = i(G)$ for every claw-free graph [44].

In another context, Cockayne, Dreyer, and Hedetniemi [19] investigated Roman graphs those satisfying $\gamma_R(G) = 2\gamma(G)$ —and showed that this equality occurs if and only if G admits a minimum Roman dominating function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$. Hansberg and Volkmann [39] characterized notable classes of graphs satisfying $\gamma(G) = \gamma_2(G)$, including cactus graphs and claw-free graphs. Cyman et al. [22] analyzed the relationship between the total domination number and the paired-domination number in k -regular graphs. They proved that $\gamma_{pr}(G)/\gamma_t(G) \leq 2k/(k+1)$ and characterized the graphs that attain this bound for specific values of k .

A related area of study concerns upper domination parameters, which refer to the maximum cardinality of a minimal dominating set of a specified type. Samodivkin [102] introduced the upper connected domination number, denoted $\Gamma_c(G)$. Ulatowski [111] investigated the upper paired-domination number, $\Gamma_p(G)$, and established the sharp bound $\Gamma_p(G) \leq n - 1$ for a connected graph G of order $n \geq 3$. Dorbec, Henning, and McCoy [27] examined relationships between upper domination parameters, proving that $\Gamma_t(T) \leq \Gamma_{pr}(T)$ for every tree T without isolated vertices, while demonstrating that no analogous inequality holds for general graphs.

Another topic concerns the uniqueness of minimum dominating structures. For example, Haynes and Henning [48] characterized trees possessing a unique minimum semipaired dominating set, and Rad and Liu [97] described trees for which the Roman domination number is equal to the unique response Roman domination number.

Following the seminal paper [26], **Paper A** [25] extends the study of this parameter by investigating the equality $\gamma(G) = \gamma_{\text{cer}}(G)$ and by introducing the upper certified domination number. Subsequent research has applied these findings in several related contexts. Navamani et al. [88–90] apply the characterization of vertices with at least two neighbors outside a dominating set to study certified subdivision, reinforcement, and bondage numbers. Cabrera-Martínez [15] examines trees satisfying $\gamma(G) = \gamma_{\text{cer}}(G)$. Anitha et al. [2] apply certified domination to shadow distance graphs. Ilyass and Kumar [60] develop the theory of connected certified domination, and Ilyass et al. [61] investigate its stability under edge addition. Additional applications are found in studies of subdivision graphs, product graphs, and criticality questions [1, 35, 78, 98, 99].

Paper A [25] investigates the relationship between the certified domination number and the standard domination number, and introduces the upper certified domination number. Its main contributions include structural conditions for the equality $\gamma(G) = \gamma_{\text{cer}}(G)$, as well as a characterization of graphs in which the certified domination number equals the order of the graph.

We first focus on the equality $\gamma(G) = \gamma_{\text{cer}}(G)$.

Theorem 2.1 ([25, Theorem 2.1]). *Let G be a connected graph of order at least three. Then $\gamma(G) = \gamma_{\text{cer}}(G)$ if and only if G has a γ -set D such that every vertex in D has at least two neighbors in $V_G - D$.*

This theorem characterizes the equality $\gamma(G) = \gamma_{\text{cer}}(G)$ via the existence of a γ -set with a prescribed neighborhood property. The following corollary provides a direct consequence for graphs with a minimum degree of at least two.

Corollary 2.2 ([25, Corollary 2.3]). *Let G be a graph with minimum degree $\delta(G) \geq 2$. Then:*

- (a) G has a γ -set D in which every vertex has at least two neighbors in $V_G - D$;
- (b) $\gamma(G) = \gamma_{\text{cer}}(G)$.

The equality $\gamma(G) = \gamma_{\text{cer}}(G)$ also holds for certain graph classes defined by forbidden induced subgraphs. Recall that a graph is P_4 -free if it contains no induced subgraph isomorphic to the path P_4 on four vertices.

Theorem 2.3 ([25, Theorem 2.7]). *Let G be a connected P_4 -free graph with $G \neq K_2$. Then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

We next examine the structure of minimal certified dominating sets.

Lemma 2.4 ([25, Lemma 3.1]). *Let G be a connected graph of order at least two. If D is a minimal certified dominating set of G and $v \in V(G)$ is a vertex whose closed neighborhood $(N_G[v])$ is contained in D , then v is either a leaf or a weak support vertex of G .*

Finally, we consider the extremal case $\gamma_{\text{cer}}(G) = n(G)$, and characterize the graphs for which this equality holds, namely those in which every nontrivial component is a corona graph.

Theorem 2.5 ([25, Theorem 3.2]). *Let G be a graph of order n . The following statements are equivalent:*

- (a) *Every non-trivial component of G is a corona;*
- (b) $\gamma_{\text{cer}}(G) = n$;
- (c) $\Gamma_{\text{cer}}(G) = n$.

2.2 A Characterization of Certified Domination Vertex-Critical Graphs (Paper B [80])

The study of domination-critical graphs traces back to the work of Sumner and Blitch [108], who introduced vertex-critical and edge-critical graphs in the context of the domination number. Specifically, a graph G is said to be k - γ -edge-critical if $\gamma(G) = k$ and the addition of any edge from the complement of G decreases the domination number to $k - 1$. Sumner and Blitch also characterized all 2- γ -edge-critical graphs. The vertex-deletion analog, called *vertex-domination-critical graphs*, was studied by Brigham, Chinn, and Dutton [13], who provided a complete characterization of the 2-vertex-critical case. A local criterion for vertex criticality was later formulated by Sampathkumar and Neeralgi [103], describing critical vertices in terms of private neighborhoods. A systematic account of these concepts was given by Sumner [107].

The notion of criticality has since been extended to various other domination parameters. For total domination, the earliest results were obtained by Cockayne, Dawes, and Hedetniemi [18], while Hansberg, Henning, and Volkmann [58] later studied further structural properties of γ_t -critical graphs. Related investigations have been carried out for paired domination [52], independent domination [33], and connected domination [14]. In addition to criticality, stability is an important concept, describing the behavior of a domination parameter under graph modifications that leave its value unchanged. In the context of certified domination, Pushpam, Kamalam, and Mahavir [94] analyzed the stability

of the certified domination number under edge addition. A comprehensive overview of domination parameters, criticality, and related concepts can be found in [46].

Paper B [80] investigates the effect of vertex deletion on the certified domination number. A vertex v is *certified critical* if

$$\gamma_{\text{cer}}(G - v) < \gamma_{\text{cer}}(G),$$

and a graph G is *vertex certified domination critical* (or γ_{cer} -critical) if every vertex of G is certified critical. In this way, **Paper B** [80] extends the classical theory of domination-critical graphs to the certified domination parameter.

This section has two main objectives. First, we present local conditions that characterize certified critical vertices. Second, we use these conditions to derive structural characterizations of γ_{cer} -critical graphs. The section concludes with a complete classification of 2-vertex certified domination critical graphs.

We begin by describing shadowed vertices in terms of private neighborhoods.

Lemma 2.6 ([80, Lemma 3.7]). *Let G be a graph. A vertex v is shadowed if and only if $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set D_{cer} of G containing v .*

The next result establishes that shadowed vertices are always certified critical.

Lemma 2.7 ([80, Lemma 3.9]). *Let G be a graph. If v is a shadowed vertex of G , then v is certified critical.*

For support vertices, this condition is both necessary and sufficient.

Lemma 2.8 ([80, Lemma 3.11]). *Let G be a graph. A support vertex s is certified critical if and only if s is shadowed.*

Another sufficient condition for certified criticality occurs when the private neighborhood of a vertex consists solely of the vertex itself.

Lemma 2.9 ([80, Lemma 3.12]). *Let G be a graph. If $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v , then v is certified critical.*

These local criteria naturally lead to structural characterizations of γ_{cer} -critical graphs. In particular, when leaves are present, the characterization takes the following form.

Corollary 2.10 ([80, Corollary 3.15 & 3.16]). *Let G be a connected graph with at least one leaf. Then G is a vertex certified domination critical graph if and only if G is a corona of some graph G' . Consequently, a tree T is γ_{cer} -critical if and only if it is the corona of some tree T' .*

For graphs with minimum degree at least two, the characterization is expressed using the condition from Lemma 3.12.

Theorem 2.11 ([80, Theorem 3.13]). *Let G be a graph with minimum degree of at least two (i.e., $\delta(G) \geq 2$). Then G is a vertex certified domination critical graph if and only if, for every vertex v of G , $pn[v, D_{cer}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v .*

Combining the leaf and minimum-degree cases gives a general structural description for all graphs.

Theorem 2.12 ([80, Theorem 3.14]). *Let G be a graph. Then G is vertex certified domination critical if and only if, for every vertex v of G , either $pn[v, D_{cer}] = \{v\}$ or $pn[v, D_{cer}] = \emptyset$ for some γ_{cer} -set D_{cer} containing v .*

The final topic in this section concerns 2-vertex certified domination critical graphs, that is, graphs G satisfying $\gamma_{cer}(G) = 2$ and

$$\gamma_{cer}(G - v) < \gamma_{cer}(G) \quad \text{for every } v \in V(G).$$

The first step in the characterization is a regularity condition on the graph.

Lemma 2.13 ([80, Lemma 5.3]). *Let G be a graph of order n . If G is a 2-vertex certified domination critical graph, then G is $(n - 2)$ -regular, and n is even.*

Using this regularity property, we can give a complete classification of such graphs.

Theorem 2.14 ([80, Theorem 5.4]). *A graph G of order n is a 2-vertex certified domination critical graph if and only if $G = K_2$, $G = \overline{K_2}$, or G is a complete graph K_{2n} with a 1-factor removed, for $n \geq 1$.*

2.3 A Characterization of $\mathcal{DD}_2$ -Graphs (Paper D [84])

Ore [91] proved that every graph without isolated vertices can be partitioned into two dominating sets. This result initiated the study of graph classes defined by vertex partitions that satisfy prescribed domination conditions. One notable example is the inverse domination number, introduced by Kulli and Sigarkanti [71], which is defined as the minimum cardinality of a dominating set whose complement also contains a dominating set.

A further direction in this area concerns partitions of the vertex set into subsets that satisfy different domination requirements. In particular, Henning and Rall [54] introduced the class of $\mathcal{DD}_2$ -graphs, defined as graphs whose vertex set can be partitioned

into a dominating set D and a 2-dominating set $V(G) \setminus D$. They established several sufficient conditions for the existence of such partitions and showed that graphs with minimum degree at least two often belong to this class in many well-known graph families.

Related partition problems have been studied for other domination parameters. For example, Henning and Southey [55] gave a constructive characterization of graphs whose vertex set can be partitioned into a dominating set and a total dominating set. Similar approaches have been applied to graphs admitting a partition into a dominating set and a paired-dominating set [56, 105, 106]. Further results were obtained by Henning and Topp [57], who studied graphs with disjoint 2-dominating sets and showed, for instance, that every bipartite graph with minimum degree at least two can be partitioned into two dominating sets.

Although these classes are closely related, they are not identical. For example, the cycle C_5 is a $\mathcal{DD}_2$ -graph, but it does not admit a partition into two dominating sets [57]. From an algorithmic perspective, such partition problems are often difficult. In particular, recognizing graphs that admit two disjoint total dominating sets is NP-complete [51], although this class also allows for a constructive characterization [53].

Paper D [84] extends the study of $\mathcal{DD}_2$ -graphs initiated by Henning and Rall [54]. Its main contribution is a complete characterization of $\mathcal{DD}_2$ -graphs for all graphs without isolated vertices, including those with vertices of degree one. The paper also provides a full description of connected minimal $\mathcal{DD}_2$ -graphs, a concept originally proposed by Jerzy Topp. These results have been applied in subsequent research on disjoint 2-domination [57], approximation problems for minimal and maximal spanning $(1, 2)$ -graphs [100], disjoint paired-dominating sets [56], and 2-domination polynomials of cycles [87].

This section has two objectives. First, we present a characterization of $\mathcal{DD}_2$ -graphs in terms of a local condition on weak support vertices. Second, we describe the structure of connected minimal $\mathcal{DD}_2$ -graphs.

We begin with the characterization of $\mathcal{DD}_2$ -graphs.

Theorem 2.15 ([84] Theorem 2.4). *Let G be a graph with no isolated vertices. The following statements are equivalent:*

1. G has a certified dominating set containing no shadowed vertex.
2. G is a $\mathcal{DD}_2$ -graph.
3. For every weak support vertex s of G , $N_G(s) - (L_G \cup S_G) \neq \emptyset$.

This theorem characterizes $\mathcal{DD}_2$ -graphs in terms of a local property of weak support vertices. In particular, it provides a linear-time criterion for recognizing graphs in this class.

Next, we consider connected minimal $\mathcal{DD}_2$ -graphs, defined as connected $\mathcal{DD}_2$ -graphs that cease to belong to the class if any edge is removed.

Theorem 2.16 ([84] Theorem 2.6). *A connected graph G is a minimal $\mathcal{DD}_2$ -graph if and only if G is a star $K_{1,n}$ with $n \geq 2$, a cycle C_4 , or the subdivision graph $S(H)$ of a connected corona multigraph H .*

2.4 Equality of Domination Numbers in Minimal $\mathcal{DD}_2$ and Bipartite Graphs (Papers B [80] and C [81])

For bipartite graphs, domination problems are closely tied to the structure of the bipartition. Let $G = (A, B)$ be a bipartite graph with $|A| \leq |B|$. Then the smaller partite set A is a natural candidate for a minimum dominating set or a minimum certified dominating set (MCDS). This perspective traces back to Cockayne and Hedetniemi [21]. In particular, since A is a dominating set, A contains a minimal dominating set, and hence $\gamma(G) \leq |A|$.

At the same time, many domination problems remain computationally hard even for bipartite graphs. For example, the standard domination problem is NP-complete for bipartite graphs, and similar complexity results are known for several related parameters [46]. In particular, paired domination [17], independent domination [30], vertex-edge domination [73], and total k -rainbow domination [64] are NP-complete for various classes of bipartite graphs.

Some problems, however, become tractable in restricted subclasses. For instance, paired domination and Roman domination can be solved in polynomial time in convex bipartite graphs [92, 101]. Other aspects of domination in bipartite graphs include conjectural upper bounds, such as Kostochka's conjecture on cubic bipartite graphs [69], and relationships with other parameters, for example, the identity $\gamma_t(G \square K_2) = 2\gamma(G)$ established in [110].

The results of **Paper C** [81] are motivated by the connection between $\mathcal{DD}_2$ -graphs and certified domination. Indeed, if G has a $\mathcal{DD}_2$ -pair $(D, V(G) \setminus D)$, then D is a certified dominating set. However, the existence of such a partition does not guarantee the equality $\gamma(G) = \gamma_{\text{cer}}(G)$, since D need not be of minimum cardinality. **Paper C** [81] focuses on this equality, first within connected minimal $\mathcal{DD}_2$ -graphs and then within connected bipartite graphs. Its starting point is the structural characterization of minimal $\mathcal{DD}_2$ -graphs

established in **Paper D** [84], while **Paper B** [80] provides related results on certified critical vertices that are applied in the bipartite case.

This section combines results from **Papers B** and **C** [81] and [80]. Its first objective is to characterize connected minimal $\mathcal{DD}_2$ -graphs satisfying $\gamma(G) = \gamma_{\text{cer}}(G)$. The second objective is to describe connected bipartite graphs in which the smaller partite set forms both a γ -set and a γ_{cer} -set. Finally, the section presents a constructive characterization of all trees for which the smaller partite set is an MCDS.

We begin with a general equivalence for $\mathcal{DD}_2$ -graphs.

Theorem 2.17 ([81, Theorem 2.1]). *Let G be a graph with no isolated vertex. The following statements are equivalent:*

1. $\gamma(G) = \gamma_{\text{cer}}(G)$;
2. *There exists a γ -set D of G such that its complement $V_G - D$ is 2-dominating.*

This theorem shows that, within the class of $\mathcal{DD}_2$ -graphs, the equality $\gamma(G) = \gamma_{\text{cer}}(G)$ is equivalent to the existence of a γ -set whose complement is 2-dominating. We now focus on connected minimal $\mathcal{DD}_2$ -graphs. Since these graphs were characterized in [84], the remaining task is to identify which of them satisfy this equality.

Lemma 2.18 ([81, Lemma 2.2]). *Let G be a minimal $\mathcal{DD}_2$ -graph of the form $G = S(\text{cor}(H))$, where H is connected. If $\gamma(G) = \gamma_{\text{cer}}(G)$, then the set $D_{\text{sub}} = \{n_e : e \in E_{\text{cor}(H)}\}$ is a γ_{cer} -set of G .*

The next lemma shows that this equality fails in the main family of minimal $\mathcal{DD}_2$ -graphs whenever the underlying graph H contains a vertex of degree at least two.

Lemma 2.19 ([81, Lemma 2.4]). *Let $G = S(\text{cor}(H))$ be a minimal $\mathcal{DD}_2$ -graph. If the maximum degree of H satisfies $\Delta(H) \geq 2$, then $\gamma(G) < \gamma_{\text{cer}}(G)$.*

These results lead to a complete characterization of connected minimal $\mathcal{DD}_2$ -graphs satisfying $\gamma(G) = \gamma_{\text{cer}}(G)$.

Theorem 2.20 ([81, Theorem 2.5]). *Let G be a connected minimal $\mathcal{DD}_2$ -graph. Then, $\gamma(G) = \gamma_{\text{cer}}(G)$ if and only if G is one of the following: a cycle C_4 ; a star $K_{1,n}$ (with $n \geq 2$); or a subdivision of $\text{cor}(K_1)$ or $\text{cor}(K_2)$.*

We now consider connected bipartite graphs. The following theorem establishes necessary and sufficient conditions under which the smaller partite set serves as both a minimum dominating set and an MCDS.

Theorem 2.21 ([81, Theorem 3.4]). *Let $G = ((A, B); E_G)$ be a connected bipartite graph of order at least three with $1 \leq |A| \leq |B|$. Then A is a γ -set and a γ_{cer} -set of G if and only if the following conditions hold:*

- (a) $S_G \cap B = \emptyset$, that is, all support vertices belong to A ;
- (b) G is a $\mathcal{DD}_2$ -graph;
- (c) For any $x, y \in A - S_G$ with $d_G(x, y) = 2$, there exist at least two vertices in B whose neighborhood is exactly $\{x, y\}$.

The proofs of the bipartite results rely on the concept of certified critical vertices developed in **Paper B** [80]. The following results establish necessary conditions for a partite set to be an MCDS.

Corollary 2.22 ([80, Corollary 4.2]). *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If A is a γ_{cer} -set of G , then A contains no shadowed vertices. Consequently, all support vertices of G belong to A (i.e., $S_G \subseteq A$).*

Theorem 2.23 ([80, Theorem 4.3]). *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If A is a γ_{cer} -set of G , then the following conditions hold:*

- (a) Every support vertex of G belongs to A ;
- (b) Every non-support vertex in A is certified critical.

The final part of this section concerns trees. Here, we employ constructive characterization, which is a standard method in domination theory. In this approach, a graph family is defined recursively by specifying an initial set of graphs together with a finite collection of graph operations. Constructive characterizations have been successfully applied to several domination-related classes, including trees with a unique minimum total dominating set [47], trees with equal paired and double domination numbers [10], and graphs in which the domination number equals the covering number [84]. Similar recursive methods are also used in the study of graphs in which a minimum dominating set covers all edges [40].

Following this approach, **Paper B** [80] gives a constructive characterization of the family $\mathcal{T}_{\text{max}}$, consisting of all trees for which the smaller partite set is an MCDS. The recursive family $\mathcal{T}$ is defined as follows. The initial graph is a star $T_0 \cong S_{1,n}$ with $n \geq 2$. Each subsequent tree T_i is obtained from T_{i-1} by one of the following operations, where (A', B') denotes the bipartition of T_{i-1} :

- *Operation $\mathcal{O}_1$* : Add a new vertex b and the edge $a'b$ for some existing vertex $a' \in A'$.
- *Operation $\mathcal{O}_2$* : Add two new adjacent vertices, a and b , and the edge ab' for some existing vertex $b' \in B'$, provided that b' is not a certified critical vertex in T_{i-1} .

The first step is to verify that every tree constructed in this manner belongs to $\mathcal{T}_{\max}$.

Lemma 2.24 ([80, Lemma 4.4]). *If a tree T belongs to the family $\mathcal{T}$, then $T \in \mathcal{T}_{\max}$.*

The converse inclusion establishes the desired constructive characterization.

Theorem 2.25 ([80, Theorem 4.5]). *A tree T belongs to $\mathcal{T}_{\max}$ if and only if $T \in \mathcal{T}$.*

2.5 Algorithmic and Computational Analysis

This section collects algorithmic and computational results on certified domination obtained in **Papers E** [95] and **F** [96]. The discussion is divided into three parts. The first part concerns exact methods, including a linear-time algorithm for trees and an integer linear programming (ILP) formulation for general graphs. The second part focuses on computational complexity, particularly the difficulty of deciding whether the domination number and the certified domination number are different. The final part considers approximation methods and compares two approaches for constructing certified dominating sets.

2.5.1 Algorithmic Aspects of Certified Domination (Paper E [95])

The algorithmic study of domination typically centers around the minimum dominating set problem: Given a graph G , find a dominating set of minimum cardinality. The decision version of this problem is NP-complete [32, 66]. For this reason, much of the literature has focused on either restricted graph classes or alternative exact and approximation methods. For example, trees admit linear-time algorithms [20, 75], and polynomial-time algorithms exist for several classes with strong structural restrictions, such as interval and circular-arc graphs [6, 16, 68]. This contrast between general graphs and structurally restricted classes has also motivated algorithmic approaches based on tree decompositions and bounded treewidth [11, 109].

Another standard approach is to formulate the domination problem as an ILP. In the classical ILP formulation, a binary variable x_v is associated with each vertex $v \in V(G)$,

where $x_v = 1$ means that v is included in the dominating set. The objective is to minimize the total number of selected vertices, $\sum_{v \in V(G)} x_v$, subject to the domination constraints

$$\sum_{u \in N_G[v]} x_u \geq 1 \quad \text{for every } v \in V(G),$$

which ensure that every vertex is dominated by at least one selected vertex.

Such ILP formulations are useful not only for exact computation but also as a basis for approximation algorithms and other algorithmic refinements [34, 36, 59, 115]. Similar approaches appear in studies of domination variants such as total domination and connected domination [29, 37, 70, 114].

In the context of certified domination, the algorithmic study is more recent. Jakkepalli, Arumugam, Khandelwal, and Reddy [63] proved that the certified domination problem is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs, and planar graphs. At the same time, they showed that the problem is solvable in linear time for chain graphs, threshold graphs, and graphs of bounded treewidth. These results place certified domination within the general complexity framework of domination theory and demonstrate that the certification condition preserves algorithmic difficulty for several important graph classes.

Paper E [95] focuses on the exact computation of the certified domination number. Its main contribution is a linear-time algorithm that finds an MCDS in a tree. Additionally, the paper presents a corrected ILP formulation for arbitrary graphs and proves its correctness, providing an exact method for computing the certified domination number on general inputs.

These results have also been used in subsequent work. For instance, Kaviya et al. [67] cite **Paper E** in the context of new certified domination parameters, including the triple-connected certified domination number.

In summary, **Paper E** [95] makes two key contributions: (i) a linear-time algorithm for computing an MCDS in trees, and (ii) a correct ILP formulation for arbitrary graphs, complete with a proof of correctness and computational verification.

2.5.1.1 Linear-Time Algorithm for Minimum Certified Dominating Set in Trees

The MCDS algorithm computes an MCDS for a given tree T in linear time. The algorithm proceeds in three phases, which are described in detail in the appendix. Each vertex v maintains a collection of local variables, initialized to zero, which are updated as the algorithm traverses the tree from the leaves toward the root.

Phase 0: Initialization. In this phase, the structural role of each vertex is identified. Based on its degree and the degrees of its neighbors, each vertex is classified as one of the following: a leaf (status L), a weak support vertex (status SW), a strong support vertex (status S), or an ordinary vertex.

Phase 1: Status Determination. This phase processes all non-leaf vertices to determine their potential inclusion in the eventual certified dominating set. Each vertex v is assigned a status (**sta**) from the set $\{0, 1, 11, 2, 22, S, SW\}$. A status of 1 indicates that v is dominated by one of its children, while a status in $\{2, 22, S, SW\}$ indicates that v is a candidate for inclusion in the final MCDS. A status of 0 signifies that other vertices are sufficient to handle local domination. The statuses 11 and 22 are intermediate states signaling specific structural configurations to be resolved in Phase 2. The status assignment is based on the statuses of v 's children, which are tracked using local counters (e.g., **n0**, **n1**, **n2**). For example, a weak support vertex may be reclassified as strong (status S) if it has non-leaf children that can provide the necessary certificates.

Phase 2: Finalization. This phase resolves any remaining ambiguities from Phase 1. It handles configurations in which a vertex might become uncertified due to the status of its parent. For instance, if a weak support vertex v (status SW) has a parent whose status requires v to be certified, a local adjustment is made by adding v 's leaf child to the set. Once all such adjustments are performed, the final minimum certified dominating set D is constructed from all vertices with a final status in $\{2, 22, S, SW\}$.

Each phase of the MCDS algorithm involves a single traversal of the tree, ensuring that the entire algorithm runs in linear time.

Theorem 2.26 ([95, Theorem 1]). *The MCDS algorithm, applied to a rooted tree T with $n(T) > 2$ and a root that is not a leaf, correctly finds a minimum certified dominating set of T .*

2.5.1.2 Integer Linear Programming for Minimum Certified Dominating Sets

This section presents a validated ILP formulation for computing a minimum certified dominating set. An alternative ILP was proposed in [63], but it is flawed: for example, it fails to produce a valid certified dominating set for the path graph P_4 . The correct ILP formulation is as follows: For each vertex $i \in V$, define a binary decision variable x_i such that $x_i = 1$ if vertex i is included in the dominating set D , and $x_i = 0$ otherwise. Additionally, introduce a binary auxiliary variable a_i for each vertex to enforce the certified condition. The objective is to minimize the function $Z = \sum_{i=1}^n x_i$, representing the cardinality of D , subject to the following constraints:

$$\forall i \in V, x_i + \sum_{j \in N(i)} x_j \geq 1 \quad (\text{C1})$$

$$\forall i \in V, \sum_{j \in N(i)} (1 - x_j) + |V| \cdot a_i \geq 2x_i \quad (\text{C2})$$

$$\forall i \in V, \sum_{j \in N(i)} (1 - x_j) + |V|(a_i - 1) + 2 \leq 2x_i \quad (\text{C3})$$

$$\forall i \in V, x_i \in \{0, 1\}, a_i \in \{0, 1\} \quad (\text{C4})$$

Constraint (C1) ensures that every vertex is either in D or adjacent to a vertex in D , guaranteeing that D is a dominating set. Constraints (C2) and (C3) together enforce the certified condition, which requires that each vertex in D have either zero or at least two neighbors outside D . Finally, (C4) enforces the binary nature of the decision variables.

Proposition 2.27 ([95, Proposition 2]). *Any feasible solution $(x, a) \in \Omega$, where $\Omega = \{(x, a) : x \in \{0, 1\}^n, a \in \{0, 1\}^n \text{ satisfy (C1), (C2), and (C3)}\}$, corresponds to a certified dominating set.*

Proposition 2.28 ([95, Proposition 3]). *Conversely, any certified dominating set D can be represented by a solution $(x, a) \in \Omega$.*

2.5.2 Complexity and Approximation of Certified Domination (Paper F [96])

The NP-hardness of the minimum dominating set problem has led to extensive research on approximation algorithms. For general graphs, standard approaches—such as greedy heuristics or linear programming relaxations—achieve approximation ratios of $O(\log n)$ or $O(\log \Delta)$, where Δ is the maximum degree. These logarithmic factors are asymptotically tight in the worst case.

For certain restricted graph classes, stronger approximation guarantees are possible. Notably, planar graphs admit a polynomial-time approximation scheme [3], providing near-optimal solutions efficiently.

In the context of certified domination, the algorithmic complexity of computing a minimum certified dominating set has been studied in earlier work. In particular, Jakkepalli et al. [63] showed that the decision problem is NP-complete for several graph classes, whereas it is solvable in polynomial time for certain restricted classes.

Paper F [96] considers a related but distinct problem. Instead of computing the exact value of $\gamma_{\text{cer}}(G)$, it investigates the complexity of deciding whether $\gamma(G)$ and $\gamma_{\text{cer}}(G)$

differ. Additionally, the paper compares two approximation strategies for constructing certified dominating sets.

The motivation for **Paper F** [96] is motivated by applications in network analysis. In particular, certified dominating sets provide an additional layer of control in social networks, thereby ensuring robustness or redundancy in influence or monitoring tasks. The results of this work have also been cited in later studies on social network analysis and domination-based models of network structure [23, 65].

The first main result of **Paper F** [96] establishes a hardness theorem, showing that comparing the two domination parameters is computationally difficult even under strong structural restrictions.

Theorem 2.29 ([96, Theorem 1]). *It is NP-hard to determine whether $\gamma(G) \neq \gamma_{\text{cer}}(G)$, even when G has only a single weak support vertex.*

This theorem demonstrates that deciding equality between $\gamma(G)$ and $\gamma_{\text{cer}}(G)$ is computationally hard, even in highly restricted graph classes. In particular, it implies that no general polynomial-time procedure is likely to exist for transforming a minimum dominating set into an MCDS.

The second part of **Paper F** [96] addresses approximation methods for constructing certified dominating sets. Two different approaches are considered: (i) transforming an existing dominating set into a certified dominating set, and (ii) constructing a certified dominating set from scratch. In this context, the paper revisits the transformation method from [63], identifies cases where the previously proposed procedure fails, and introduces a corrected algorithm, denoted **ClassicToCert**. Additionally, a second algorithm, **ApproxCert**, is proposed to build a certified dominating set directly.

The **ClassicToCert** algorithm, described in the appendix, provides a method to transform a minimal dominating set into a certified dominating set by systematically addressing vertices that violate the certification condition. First, a minimal dominating set D is computed for the graph G . The algorithm then identifies the set U of uncertified vertices, consisting of all $u \in D$ that have exactly one neighbor outside of D .

The main loop continues while U is nonempty. At each iteration, a vertex u is chosen from U , along with its unique neighbor v that is not in D . The set D is then updated according to the structural role of u . If u is a leaf, it is removed from D and its support vertex v is added. If u is a support vertex, then if v is a leaf, v is added to D to certify u ; otherwise, all leaves adjacent to u are removed from D . If u is neither a leaf nor a support vertex, the algorithm checks whether u is the only vertex in D dominating v ; if so, u and v are swapped to maintain domination responsibility, otherwise u is simply removed.

After each modification, the set U is recalculated. The loop terminates when U is empty, at which point all vertices in D satisfy the certification condition, and D forms a valid certified dominating set.

Theorem 2.30 ([96, Theorem 2]). *Algorithm 1, `ClassicToCert`, returns a certified dominating set of G .*

The **ApproxCert** algorithm, described in the appendix, constructs a minimal certified dominating set from scratch through several phases. The first phase is initialization, where the **Initialize** function prepares the graph by identifying all support vertices and adding them to the initial dominating set D , which is necessary for any certified dominating set. During this phase, each vertex is also assigned several local variables, including `.pow`, a heuristic estimating the vertex's usefulness for dominating its neighbors. After initialization, the algorithm enters a greedy construction phase, where a while loop repeatedly adds the vertex with the highest `.pow` value to D until all vertices in the graph are dominated. Once a dominating set is formed, the algorithm proceeds to a minimization phase. In this phase, a “for” loop examines all vertices in D and removes any vertex whose neighbors are already dominated by other vertices in D , ensuring that the set is as small as possible. The final certification phase addresses any remaining issues using two separate “for” loops. The first loop handles uncertified non-support vertices by replacing them with their unique outside neighbor. The second loop addresses weak support vertices by adding their unique leaf neighbor to D . The resulting set D is guaranteed to be a minimal certified dominating set.

Theorem 2.31 ([96, Theorem 3]). *The `ApproxCert` function (Algorithm 3) finds a minimal certified dominating set in polynomial time.*

Chapter 3

Conclusion and Future Work

This dissertation investigates certified domination in graphs from both structural and algorithmic perspectives. The main parameter studied is the certified domination number, $\gamma_{\text{cer}}(G)$. The dissertation analyzes its fundamental properties and its relationship to the domination number, $\gamma(G)$. One of the main results demonstrates that the equality

$$\gamma(G) = \gamma_{\text{cer}}(G)$$

is closely related to the class of $\mathcal{DD}_2$ -graphs. Specifically, this equality holds if and only if G has a minimum dominating set D such that $V(G) \setminus D$ forms a 2-dominating set.

A substantial portion of the dissertation investigates the structural properties of certified domination. In particular, the concept of criticality is used to study γ_{cer} -critical graphs, defined as graphs in which the removal of any vertex decreases the certified domination number. For trees, this class is completely characterized and coincides with the class of corona graphs. The dissertation also presents several algorithmic contributions. It is shown that determining whether

$$\gamma(G) \neq \gamma_{\text{cer}}(G)$$

is NP-hard, even for restricted graph classes. On the positive side, a linear-time algorithm for finding an MCDS in trees is developed, and a correct ILP formulation is provided for arbitrary graphs. Two heuristic methods for constructing certified dominating sets are also compared. The computational analysis indicates that direct construction is more effective than approaches that transform an existing dominating set into a certified dominating set. These results are further illustrated through applications to fire safety water networks [95] and social network analysis [96].

The findings of this dissertation suggest several promising directions for future research. One remaining problem is to determine the necessary and sufficient conditions under

which a bipartite graph $G = ((A, B); E_G)$ satisfies

$$\gamma_{\text{cer}}(G) = \min\{|A|, |B|\}.$$

It would also be of interest to identify and characterize the graphs for which

$$\gamma_{\text{cer}}(G) = \Gamma_{\text{cer}}(G).$$

Another research direction involves studying k -vertex γ_{cer} -critical graphs for $k \geq 3$, together with the corresponding class of γ_{cer} -stable graphs. A natural extension is the concept of a k -certified dominating set, in which every vertex of the dominating set has either zero or at least k neighbors outside the set. This leads to the parameter $\gamma_{\text{cer},k}(G)$ and the corresponding class of $\mathcal{DD}_k$ -graphs.

From the algorithmic perspective, two research directions are especially promising. The first is to determine the precise approximation ratio of the algorithm **ApproxCert**. The second is to design parameterized algorithms for certified domination, with a particular focus on algorithms parameterized by treewidth [11, 109].

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Appendix A: Linear algorithm for finding Minimum Certified Dominating Set in Trees

Algorithm: Phase 0 of MinimumCertifiedDominatingSet

Data: A tree T with tree-order

Result: T with identified L , SW and S nodes

```

1 Function Phase0 of MinimumCertifiedDominatingSet( $T$ ):
2   for  $v \in V(T)$  (from the last to the root) do
3     if  $v$  is of degree 1 then
4        $v.sta \leftarrow L$ ;
5        $F(v).leaf \leftarrow F(v).leaf + 1$ ;
6        $F(v).add \leftarrow v$ ;
7     else
8       if  $v.leaf > 0$  then
9         if  $v.leaf = 1$  then  $v.sta \leftarrow SW$  else  $v.sta \leftarrow S$ ;
10    return  $T$ 

```

Algorithm: Phase 1 of MinimumCertifiedDominatingSet

Data: A tree T with tree-order and identified L , SW and S nodes

Result: T with determined statuses

```

1 Function Phase1 of MinimumCertifiedDominatingSet( $T$ ):
2   for  $v \in V(T) - L(T)$  (from the last to the root) do
3     if  $v.leaf > 0$  then
4       if  $v.n_0 + v.n_1 + v.n_{22} = 0$  and  $v.leaf = 1$  then  $F(v).sw \leftarrow F(v).sw + 1$ ;
5       else  $v.sta \leftarrow S$ ;
6        $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
7     else
8       if  $v.n_0 = 0$  then
9         if  $v.n_2 > 0$  then
10           $v.sta \leftarrow 1$ ;
11           $F(v).n_1 \leftarrow F(v).n_1 + 1$ ;
12        else
13           $v.sta \leftarrow 0$ ;
14           $F(v).n_0 \leftarrow F(v).n_0 + 1$ ;
15        else
16          if  $v.n_0 = 1$  and  $v.n_1 + v.sw = 0$  then
17             $v.sta \leftarrow 22$ ;
18             $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
19             $F(v).n_{22} \leftarrow F(v).n_{22} + 1$ ;
20          else
21            if  $F(v).sta = SW$  then  $f = 1$  else  $f = 0$ ;
22            if  $v.n_0 > v.sw + f$  then
23               $v.sta \leftarrow 2$ ;
24               $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
25            else
26               $v.sta \leftarrow 11$ ;
27               $F(v).n_1 \leftarrow F(v).n_1 + 1$ ;
28   if  $root.n_2 = 0$  then  $root.sta \leftarrow 2$ ;
29   return  $T$ 

```

Algorithm: Phase 2 of MinimumCertifiedDominatingSet

Data: A tree T with determined statuses

Result: T with determined minimum certified dominating set

```

1 Function Phase2( $T$ ):
2   for  $v \in V(T) - L(T)$  (from last to the root) do
3     if  $v$  is not the root then
4       if  $v.sta = SW$  and  $F(v).sta \in \{2, SW, S\}$  then  $(v.add).sta \leftarrow 2$ ;
5       if  $v.sta = 0$  and  $F(v).sta = 11$  then  $v.sta \leftarrow 2$ ;
6       if  $v.sta = 0$  and  $F(v).sta = 22$  and  $F(F(v)).sta \in \{2, 22, SW, S\}$  then
7          $v.sta \leftarrow 2$ ;
8          $F(v).sta \leftarrow 1$ ;
9         if  $F(F(v)).sta = SW$  then  $F(F(v)).sta \leftarrow S$ ;
10    if  $root.sta = SW$  and  $root.n_0 + root.n_1 = 0$  then  $(root.add).sta \leftarrow 2$ ;
11     $D \leftarrow \{v \in V(T) : v.sta \in \{2, 22, S, SW\}\}$ ;
12    return  $D$ 

```

Appendix B: ClassicToCert algorithm

Algorithm: ClassicToCert function
Data: A simple graph G
Result: A certified dominating set D

```

1  $D \leftarrow$  a minimal dominating set of  $G$ ;
2  $U \leftarrow \{u \in D : |N(u) \cap (V(G) \setminus D)| = 1\}$ ;
3 while  $U \neq \emptyset$  do
4   Let  $u \in U$  and let  $v$  be the unique vertex in  $N(u) \setminus D$ ;
5   if  $u$  is a leaf then
6      $D \leftarrow (D \cup \{v\}) \setminus \{u\}$ ;
7   else
8     if  $u$  is a support vertex then
9       if  $v$  is a leaf then
10         $D \leftarrow D \cup \{v\}$ ;
11      else
12        Remove from  $D$  all leaves adjacent to  $u$ ;
13    else
14      if  $N[v] \cap D = \{u\}$  then
15         $D \leftarrow (D \cup \{v\}) \setminus \{u\}$ ;
16      else
17         $D \leftarrow D \setminus \{u\}$ ;
18     $U \leftarrow \{u \in D : |N(u) \cap (V(G) \setminus D)| = 1\}$ ;
19 return  $D$ 

```

Appendix C: ApproxCert algorithm

Algorithm: Initialize function
Data: A simple graph G
Result: G with support vertices added to D

```

1  $v.sta \leftarrow L$  if and only if  $v$  is a leaf;
2  $v.sta \leftarrow S$  if and only if  $v$  is a support vertex;
3  $v.D \leftarrow 1$  if and only if  $v$  is a support vertex;
4 for  $v \in V(G)$  do
5    $v.out \leftarrow |\{y \in N(v) : y.D = 0\}|$ ;
6    $v.dom \leftarrow |\{y \in N[v] : y.D = 1\}|$ ;
7 for  $v \in V(G)$  do
8    $v.pow \leftarrow |\{y \in N[v] : y.dom = 0\}|$ ;
9   for  $y \in N[v]$  do
10    if  $y.D = 1$  and  $y.out = 2$  then
11       $v.pow \leftarrow v.pow - 1$ ;
12    if  $v.out = 1$  then
13       $v.pow \leftarrow v.pow - 1$ ;
14 return  $G$ 

```

Algorithm: ApproxCert function
Data: A simple graph G
Result: A certified dominating set D

```

1  $G \leftarrow \text{Initialize}(G);$ 
2 while  $\exists v \in V(G)$  such that  $v.dom = 0$  do
3    $x \leftarrow$  a vertex with the maximum value of  $pow$ ;
4    $x.D \leftarrow 1$ ;
5   Update  $y.dom$  and  $y.out$  for all  $y \in N[x]$ ;
6   Update  $y.pow$  for all  $y \in N_2[x]$ ;
7 for  $v \in V(G)$  do
8   if  $v.D = 1$  and  $v.dom > 1$  and  $\forall y \in N(v)$  ( $y.D = 1$  or  $y.dom > 1$ ) then
9      $v.D \leftarrow 0$ ;
10    Update  $y.dom$  and  $y.out$  for all  $y \in N[v]$ ;
11 for  $v \in V(G)$  do
12   if  $v.sta \neq S$  and  $v.D = 1$  and  $v.out = 1$  then
13     for  $y \in N(v)$  do
14       if  $y.D = 0$  then
15          $y.D \leftarrow 1$ ;
16          $v.D \leftarrow 0$ ;
17         Update  $z.dom$  and  $z.out$  for all  $z \in N[y] \cup N[v]$ ;
18 for  $v \in V(G)$  do
19   if  $v.sta = S$  and  $v.out = 1$  then
20     for  $y \in N(v)$  do
21       if  $y.D = 0$  then
22          $y.D \leftarrow 1$ ;
23  $D \leftarrow \{v \in V(G) : v.D = 1\};$ 
24 return  $D$ 

```

GRAPHS WITH EQUAL DOMINATION AND CERTIFIED DOMINATION NUMBERS

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Abstract. A set D of vertices of a graph $G = (V_G, E_G)$ is a dominating set of G if every vertex in $V_G - D$ is adjacent to at least one vertex in D . The domination number (upper domination number, respectively) of G , denoted by $\gamma(G)$ ($\Gamma(G)$, respectively), is the cardinality of a smallest (largest minimal, respectively) dominating set of G . A subset $D \subseteq V_G$ is called a certified dominating set of G if D is a dominating set of G and every vertex in D has either zero or at least two neighbors in $V_G - D$. The cardinality of a smallest (largest minimal, respectively) certified dominating set of G is called the certified (upper certified, respectively) domination number of G and is denoted by $\gamma_{\text{cer}}(G)$ ($\Gamma_{\text{cer}}(G)$, respectively). In this paper relations between domination, upper domination, certified domination and upper certified domination numbers of a graph are studied.

Keywords: domination, certified domination.

Mathematics Subject Classification: 05C69.

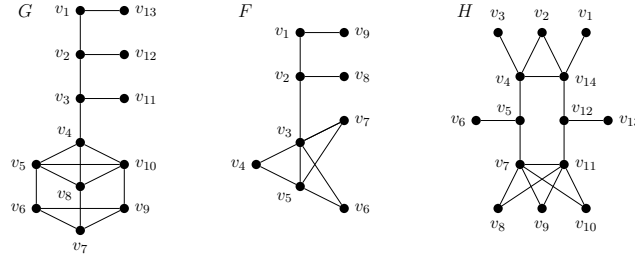
1. NOTATION AND DEFINITIONS

We generally follow the notation and terminology of [7]. For a graph G , the set of vertices is denoted by V_G and the edge set by E_G . For a vertex $v \in V_G$, the *open neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v , and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v . The *open neighborhood* of a set $X \subseteq V_G$ is $N_G(X) = \bigcup_{v \in X} N_G(v)$, whereas the *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$. For $X \subseteq V_G$ and $v \in X$, the set $N_G[v] - N_G[X - \{v\}]$ is denoted by $PN_G[v, X]$ and called the *private neighborhood* of v with respect to X . Every vertex belonging to $PN_G[v, X]$ is

called a *private neighbor* of v with respect to X . By $PN_G(v, X)$ we denote the set $N_G(v) - N_G[X - \{v\}]$ and call it the *open private neighborhood* (of v with respect to X). The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. The number $\min\{d_G(v) : v \in V_G\}$ is the minimum degree of G and is denoted by $\delta(G)$. A vertex of degree 0 is called an *isolated vertex*, whereas a vertex of degree one in G is called a *leaf* of G . If v is a leaf, then its only neighbor is called a *support vertex* of v . A support vertex is called *strong* or *weak* depending on whether or not it is adjacent to at least two leaves. We use L_G , S_G , S_G^1 and S_G^2 to denote the set of all leaves, support vertices, weak support vertices and strong support vertices of G , respectively. Finally, the *corona* $H \circ K_1$ of a graph H was defined in [4] as the graph obtained from H by adding exactly one pendant edge to each vertex of H . A graph G is said to be a *corona* if it is the corona $H \circ K_1$ of some graph H . It is obvious that a corona is a graph in which each vertex is a leaf or a weak support vertex.

Given a graph G , we say that a subset $D \subseteq V_G$ is a *dominating set* of G if every vertex belonging to $V_G - D$ is adjacent to at least one vertex in D . The *domination number* (*upper domination number*, respectively) of G , denoted by $\gamma(G)$ ($\Gamma(G)$, respectively), is the cardinality of a smallest (largest minimal, respectively) dominating set of G . A dominating (minimal dominating, respectively) set of G of minimum (maximum, respectively) cardinality is called a γ -set (Γ -set, respectively) of G . A subset $D \subseteq V_G$ is called a *certified dominating set* of G if D is a dominating set of G and every vertex belonging to D has either zero or at least two neighbors in $V_G - D$. The cardinality of a smallest (largest minimal, respectively) certified dominating set of G is called the *certified* (*upper certified*, respectively) *domination number* of G and is denoted by $\gamma_{\text{cer}}(G)$ ($\Gamma_{\text{cer}}(G)$, respectively). A certified dominating (minimal certified dominating, respectively) set of G of minimum (maximum, respectively) cardinality is called a γ_{cer} -set (Γ_{cer} -set, respectively) of G . For example, it is easy to observe that for the most common graph families, we have $\gamma(K_n) = \gamma_{\text{cer}}(K_n) = \Gamma(K_n) = \Gamma_{\text{cer}}(K_n) = 1$ if $n \neq 2$, $\gamma(P_n) = \gamma_{\text{cer}}(P_n) = \lceil n/3 \rceil$ and $\Gamma_{\text{cer}}(P_n) = \lfloor (n-1)/2 \rfloor = \Gamma(P_n) - 1$ if $n \geq 5$, $\gamma(C_n) = \gamma_{\text{cer}}(C_n) = \lceil n/3 \rceil$ and $\Gamma(C_n) = \Gamma_{\text{cer}}(C_n) = \lfloor n/2 \rfloor$ if $n \geq 3$, $\Gamma(K_{1,n}) = n$ (if $n \geq 1$) and $\gamma(K_{1,n}) = \gamma_{\text{cer}}(K_{1,n}) = \Gamma_{\text{cer}}(K_{1,n}) = 1$ if $n \geq 2$, $\gamma(K_{m,n}) = \gamma_{\text{cer}}(K_{m,n}) = 2$ and $\Gamma(K_{m,n}) = \Gamma_{\text{cer}}(K_{m,n}) = n$ if $2 \leq m \leq n$.

It is obvious that for any graph G we have the inequalities $\gamma(G) \leq \gamma_{\text{cer}}(G) \leq \Gamma_{\text{cer}}(G)$, while the parameters $\gamma_{\text{cer}}(G)$ and $\Gamma(G)$, and also the parameters $\Gamma_{\text{cer}}(G)$ and $\Gamma(G)$ are not comparable. For example, the inequalities $\gamma(G) \leq \Gamma(G) \leq \gamma_{\text{cer}}(G) \leq \Gamma_{\text{cer}}(G)$ hold for the graph G in Figure 1. In this case it is easy to check that $\gamma(G) = 5$, $\Gamma(G) = 6$, $\gamma_{\text{cer}}(G) = 7$, $\Gamma_{\text{cer}}(G) = 8$, and the sets $\{v_1, v_2, v_3, v_5, v_6\}$, $\{v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_1, v_2, v_3, v_5, v_6, v_{12}, v_{13}\}$, $\{v_1, v_2, v_3, v_5, v_8, v_{10}, v_{12}, v_{13}\}$ are examples of γ -, Γ -, γ_{cer} -, and Γ_{cer} -sets of G , respectively. The graph F in Figure 1 illustrates the inequalities $\gamma(F) \leq \gamma_{\text{cer}}(F) \leq \Gamma(F) \leq \Gamma_{\text{cer}}(F)$. One can check that $\gamma(F) = 3$, $\gamma_{\text{cer}}(F) = 4$, $\Gamma(F) = 5$, $\Gamma_{\text{cer}}(F) = 6$, and the sets $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_5, v_9\}$, $\{v_4, v_6, v_7, v_8, v_9\}$ and $\{v_1, v_2, v_4, v_6, v_7, v_9\}$ are examples of γ -, γ_{cer} -, Γ -, and Γ_{cer} -sets of F , respectively. Finally, the inequalities $\gamma(H) \leq \gamma_{\text{cer}}(H) \leq \Gamma_{\text{cer}}(H) \leq \Gamma(H)$ hold for the graph H in Figure 1, where $\gamma(H) = 5$, $\gamma_{\text{cer}}(H) = 6$, $\Gamma_{\text{cer}}(H) = 7$, $\Gamma(H) = 8$, and $\{v_4, v_5, v_7, v_{12}, v_{14}\}$, $\{v_4, v_5, v_6, v_7, v_{12}, v_{14}\}$, $\{v_4, v_5, v_8, v_9, v_{10}, v_{12}, v_{14}\}$ and $\{v_1, v_2, v_3, v_6, v_8, v_9, v_{10}, v_{13}\}$ are examples of γ -, γ_{cer} -, Γ_{cer} - and Γ -sets of H , respectively.


 Fig. 1. Graphs G , F , and H

Domination in graphs is one of the most fundamental and well-studied concepts in graph theory. The reader is referred to [6, 7] and [8] for more details on these important topics. The previously mentioned certified domination was introduced by Dettlaff *et al.* [2] in order to describe some possible relations in social networks. In this paper we continue the study of certified dominating sets and certified domination numbers of graphs. For different classes of graphs G we establish conditions for the equality of the domination number $\gamma(G)$ and the certified domination number $\gamma_{\text{cer}}(G)$ of a graph G . Furthermore, we characterize all graphs G for which $\gamma(H) = \gamma_{\text{cer}}(H)$ for each induced and connected subgraph $H \neq K_2$ of G . The last part of the paper is concerned with main properties of the upper certified domination number $\Gamma_{\text{cer}}(G)$ of G and its relations to $\gamma_{\text{cer}}(G)$ and $\Gamma(G)$. We conclude with some open problems.

2. GRAPHS G FOR WHICH $\gamma_{\text{cer}}(G) = \gamma(G)$

In this section we study basic properties which guarantee equalities of domination and certified domination numbers. We begin with the following necessary and sufficient condition for the equality of domination and certified domination numbers of a graph.

Theorem 2.1. *Let G be a connected graph of order at least three. Then $\gamma(G) = \gamma_{\text{cer}}(G)$ if and only if G has a γ -set D such that every vertex in D has at least two neighbors in $V_G - D$.*

Proof. Assume that $\gamma(G) = \gamma_{\text{cer}}(G)$. Let D be a γ_{cer} -set of G . Then the equality $\gamma(G) = \gamma_{\text{cer}}(G)$ guarantees that D is also a γ -set of G . Now, let v be a vertex in D . Since v is not an isolated vertex, the set $N_G(v) \cap (V_G - D)$ is nonempty (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Consequently, since D is a certified dominating set, we have $|N_G(v) \cap (V_G - D)| \geq 2$.

On the other hand, if D is a γ -set of G and $|N_G(v) \cap (V_G - D)| \geq 2$ for every $v \in D$, then D is also a certified dominating set of G . Hence $\gamma_{\text{cer}}(G) \leq |D| = \gamma(G) \leq \gamma_{\text{cer}}(G)$ and therefore $\gamma(G) = \gamma_{\text{cer}}(G)$. $\square$

From Theorem 2.1 we immediately obtain the following corollary.

Corollary 2.2. *Let G be a connected graph of order at least three. If G has an independent γ -set that contains no leaf of G , then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

It was already proved in [2] that $\gamma(G) = \gamma_{\text{cer}}(G)$ for all graphs G without leaves. Here we present another proof of that result.

Corollary 2.3. *If G is a graph in which $\delta(G) \geq 2$, then:*

- (1) G has a γ -set D such that every vertex in D has at least two neighbors in $V_G - D$;
- (2) $\gamma(G) = \gamma_{\text{cer}}(G)$.

Proof. For a γ -set X of G , let $q(X)$ denote the set $\{x \in X : |N_G(x) \cap (V_G - X)| \leq 1\}$. Now, let D be a γ -set such that $|q(D)|$ is as small as possible. We claim that $q(D) = \emptyset$ and so D is the required set. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex in $q(D)$. Since $d_G(v) \geq \delta(G) \geq 2$, the set $N_G(v)$ cannot be a subset of D , as otherwise $D - \{v\}$ would be a dominating set of G . Consequently $|N_G(v) \cap (V_G - D)| = 1$, say $N_G(v) \cap (V_G - D) = \{v'\}$. Again, since $d_G(v) \geq 2$ and $|N_G(v) \cap (V_G - D)| = 1$, $N_G(v) \cap D \neq \emptyset$ and v' is the only private neighbor of v with respect to D . Thus $N_G(v') - \{v\}$ is a non-empty subset of $V_G - D$ and $D' = (D - \{v\}) \cup \{v'\}$ is a minimum dominating set of G and the set

$$q(D') = \{x \in D' : |N_G(x) \cap (V_G - D')| \leq 1\}$$

is of size smaller than $|q(D)|$ (as $q(D') \subseteq q(D) - \{v\}$, since v' is a private neighbor of v with respect to D and $d_G(v') \geq 2$), a contradiction, which completes the proof of (1). The property (2) follows from (1) and Theorem 2.1. $\square$

It has been proved in [5] that if D is a unique γ -set of a graph G , then every vertex in D that is not an isolated vertex has at least two private neighbors other than itself, that is, in the set $V_G - D$. From this and from Theorem 2.1 we have the following corollary.

Corollary 2.4 ([2]). *If a graph G has a unique γ -set, then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

The main properties of graphs having unique γ -sets have been studied in [5] and partially in [3]. It was also observed in [5] that if D is a γ -set of a graph G and $\gamma(G - x) > \gamma(G)$ for every $x \in D$, then D is the unique γ -set of G . Thus, by Corollary 2.4, we have the next corollary.

Corollary 2.5. *If a graph G has a γ -set D such that $\gamma(G - x) > \gamma(G)$ for every $x \in D$, then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

The next theorem provides another sufficient condition for the equality of domination and certified domination numbers of a graph.

Theorem 2.6. *Let G be a connected graph of order at least three. If $\gamma(G - v) \geq \gamma(G)$ for every vertex v belonging to at least one γ -set of G , then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

Proof. Similarly as in the proof of Corollary 2.3, if X is a γ -set of G , then by $q(X)$ we denote the set $\{x \in X : |N_G(x) \cap (V_G - X)| \leq 1\}$. Assume that $\gamma(G - x) \geq \gamma(G)$ for every vertex x belonging to at least one γ -set of G . To prove that $\gamma(G) = \gamma_{\text{cer}}(G)$, by Theorem 2.1, it remains to show that $q(X) = \emptyset$ for some γ -set X of G .

Let D be a γ -set such that $|q(D)|$ is as small as possible. We claim that $q(D) = \emptyset$ and so D is the required set. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex belonging to $q(D)$. Since v is not an isolated vertex, the minimality of D implies that $N_G(v) \cap (V_G - D) \neq \emptyset$ (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Thus $|N_G(v) \cap (V_G - D)| = 1$, and let v' be the only vertex in $N_G(v) \cap (V_G - D)$. Then the set $D' = (D - \{v\}) \cup \{v'\}$ is a γ -set of G .

We now claim that v' is a private neighbor of v with respect to D . Suppose, contrary to our claim, that $v' \notin \text{PN}_G[v, D]$. Then $v' \in N_G(D - \{v\})$ and either $N_G(v) \cap D \neq \emptyset$ or $N_G(v) \cap D = \emptyset$. The first case is impossible (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Thus $N_G(v) \cap D = \emptyset$, but now $D - \{v\}$ is a dominating set of $G - v$ and therefore, $\gamma(G - v) \leq |D - \{v\}| < |D| = \gamma(G)$, contrary to our assumption. This proves that v' is a private neighbor of v with respect to D .

Next, since the private neighbor v' of v with respect to D is the only neighbor of v in $V_G - D$ and since G is a connected graph of order at least three, the sets $N_G(v) - \{v'\}$ and $N_G(v') - \{v\}$ are subsets of D and $V_G - D$, respectively, and at least one of them is non-empty. If $N_G(v) - \{v'\} \neq \emptyset$, then $\text{PN}_G[v', D] = \{v'\}$ and $D' - \{v'\}$ is a dominating set of $G - v'$, and then $\gamma(G - v') \leq |D' - \{v'\}| < |D'| = \gamma(G)$, contrary to our assumption. Thus assume that $N_G(v) - \{v'\} = \emptyset$. Then $N_G(v') - \{v\}$ is a non-empty subset of $V_G - D'$, and therefore

$$\begin{aligned} |N_G(v') \cap (V_G - D')| &= |\{v\}| + |(N_G(v') - \{v\}) \cap (V_G - D')| \\ &= 1 + |N_G(v') - \{v\}| \geq 2. \end{aligned}$$

This implies that $v' \notin q(D')$. Now, since $v \notin q(D')$ (as $v \notin D'$), neither v' nor v belongs to $q(D')$, and therefore $q(D') \subseteq q(D) - \{v\}$. Consequently, we have $|q(D')| < |q(D)|$, a contradiction to the choice of D , and this completes the proof. $\square$

The fundamental relations between the classes of graphs considered in Corollaries 2.3–2.5 and Theorems 2.1 and 2.6 are illustrated by examples in Figure 2.

A graph G is said to be P_4 -free if P_4 is not an induced subgraph of G . We also say that a graph G is γ_{cer} -perfect if $\gamma(H) = \gamma_{\text{cer}}(H)$ for each induced and connected subgraph $H \neq K_2$ of G . It follows from this definition that a graph G is γ_{cer} -perfect if and only if each component of G , different from K_2 , is γ_{cer} -perfect. The path P_4 is the smallest non- γ_{cer} -perfect graph. The union $K_2 \cup C_4$ is a γ_{cer} -perfect graph, while the union $K_2 \cup C_5$ is not a γ_{cer} -perfect graph. We now study the equality $\gamma(G) = \gamma_{\text{cer}}(G)$ for P_4 -free graphs.

Theorem 2.7. *If G is a connected P_4 -free graph and $G \neq K_2$, then $\gamma(G) = \gamma_{\text{cer}}(G)$.*

Proof. The result is obvious if $G = K_1$. Thus assume that G is a connected P_4 -free graph of order at least three. We shall prove that $\gamma(G) = \gamma_{\text{cer}}(G)$. Since $\gamma(G) \leq \gamma_{\text{cer}}(G)$, it suffices to show that some γ -set of G is a certified dominating set of G .

Similarly as in the proof of Corollary 2.3 and Theorem 2.6, let D be a γ -set of G such that D contains no leaf of G and $q(D) = \{x \in D : |N_G(x) \cap (V_G - D)| \leq 1\}$ is as small as possible (since G is connected and $G \neq K_2$, such D exists). We claim that $q(D) = \emptyset$. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex belonging to $q(D)$. By the same reason as in the proof of Theorem 2.6, there exists v' which is the only neighbor of v in $V_G - D$. Since v is not a leaf and v' is the only neighbor of v in $V_G - D$, the set $N_G(v) \cap D$ is non-empty. Choose any vertex $u \in N_G(v) \cap D$. The minimality of D and the fact that v and u are adjacent elements of D imply that the private neighborhoods $PN_G[v, D]$ and $PN_G[u, D]$ are disjoint and non-empty subsets of $V_G - D$. Certainly, $PN_G[v, D] = \{v'\}$ and, therefore, $N_G(x) \cap \{v'\} = \emptyset$ for every $x \in D - \{v\}$. Now, since G is a P_4 -free graph, the vertex v' is not a leaf and it is adjacent to every vertex in $PN_G[u, D]$ and, therefore, to every vertex in $\bigcup_{u \in N_G(v) \cap D} PN_G[u, D]$. Again from the fact that G is P_4 -free it is easily seen that $N_G(v) \cap D = \{u\}$ and $N_G(u) \cap D = \{v\}$ (otherwise, the set $\{u', v', v, w\}$, where $u' \in PN_G[u, D]$ and $w \in (N_G(v) \cap D) - \{u\}$, or, respectively, the set $\{v', v, u, z\}$, where $z \in (N_G(u) \cap D) - \{v\}$, would induce a 4-vertex path in G , a contradiction). Now let us consider the set $D' = (D - \{v\}) \cup \{v'\}$. It is obvious that D' is a minimum dominating set of G and D' contains no leaf of G . It remains to show that $q(D') \subseteq q(D) - \{v\}$. First, let us observe that $v \notin q(D')$ (as $v \notin D'$) and $\{v', u\} \cap q(D') = \emptyset$ (as $|N_G(x) \cap (V_G - D')| \geq |\{v\} \cup PN_G[u, D]| \geq 2$ if $x \in \{v', u\}$). Now, since $N_G(v) \cap q(D') = \{v', u\} \cap q(D') = \emptyset$, we have $N_G(x) \cap \{v\} = \emptyset$ for every $x \in q(D')$. Consequently, if $x \in q(D')$, then $x \in q(D) - \{v, v', u\}$ and therefore we have

$$\begin{aligned}
 1 &= |N_G(x) \cap (V_G - D')| \\
 &= |N_G(x) \cap (V_G - ((D - \{v\}) \cup \{v'\}))| \\
 &= |N_G(x) \cap (((V_G - D) - \{v'\}) \cup \{v\})| \\
 &= |N_G(x) \cap ((V_G - D) - \{v'\}) \cup (N_G(x) \cap \{v\})| \\
 &= |((N_G(x) \cap (V_G - D)) - (N_G(x) \cap \{v'\})) \cup (N_G(x) \cap \{v\})| \\
 &= |N_G(x) \cap (V_G - D)|,
 \end{aligned}$$

as $N_G(x) \cap \{v'\} = \emptyset$ and $N_G(x) \cap \{v\} = \emptyset$. This shows that $q(D') \subseteq q(D) - \{v\}$, a contradiction, which completes the proof. $\square$

Theorem 2.7 immediately implies the following characterization of the $\gamma\gamma_{\text{cer}}$ -perfect graphs.

Corollary 2.8. *A graph G is a $\gamma\gamma_{\text{cer}}$ -perfect graph if and only if G is a P_4 -free graph.*

Proof. The “only if” part of the theorem follows from the fact that $\gamma(P_4) = 2 < 4 = \gamma_{\text{cer}}(P_4)$. The “if” part follows from Theorem 2.7. $\square$

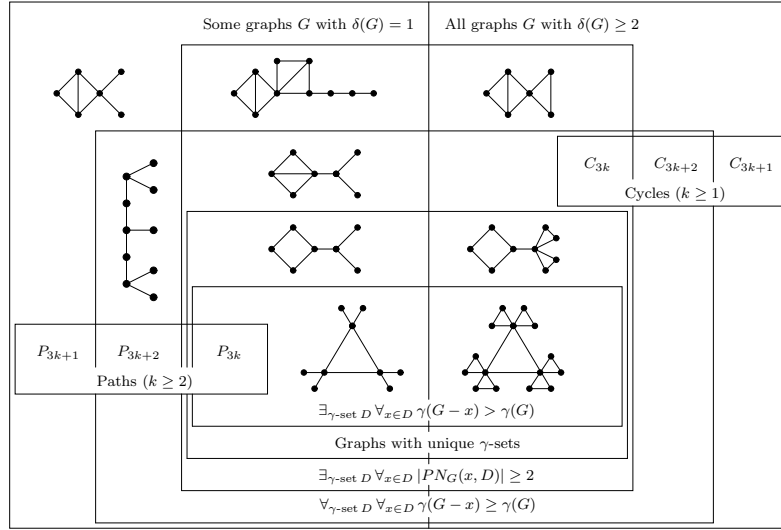


Fig. 2. The variety of relations between the classes of graphs G with $\gamma(G) = \gamma_{\text{cer}}(G)$

3. PROPERTIES OF UPPER CERTIFIED DOMINATION NUMBER

In this section we study main properties of upper certified domination number Γ_{cer} . We give a characterization of all graphs with $\Gamma_{\text{cer}} = n$ and $\Gamma_{\text{cer}} = n - 2$, respectively. In addition, we focus on the relation between upper domination number and upper certified domination number of a graph. We start with the following useful lemma.

Lemma 3.1. *Let G be a connected graph of order at least two. If D is a minimal certified dominating set of G and v is a vertex such that $N_G[v] \subseteq D$, then $v \in L_G \cup S_G^1$, that is, v is a leaf or a weak support vertex of G . In addition, the induced subgraph $G[\{v \in D : N_G[v] \subseteq D\}]$ is a corona.*

Proof. The result is obvious if $G = K_2$. Thus assume that G is a graph of order at least three. Suppose, contrary to our claim, that the set

$$D' = \{v \in D : N_G[v] \subseteq D\} - (L_G \cup S_G^1)$$

is non-empty. Let F denote the subgraph $G[D']$. Let I be a maximal set of independent vertices of degree at least two in F . The set I is a proper (possibly empty) subset of D' and, if I is non-empty, then every vertex in I has at least two neighbors in $D' - I$. Now let I' denote the set $D' - N_F[I]$. We claim that I' is dominated by the set $D - (L_G \cup D')$.

This is trivially true if $I' = \emptyset$. Thus assume that $I' \neq \emptyset$ and $v_0 \in I'$. Then $d_G(v_0) \geq 2$ (as $v_0 \notin L_G$) but $d_F(v_0) \leq 1$ (otherwise $I \cup \{v_0\}$ would be a larger set of independent vertices of degree at least two in F) and therefore $N_G(v_0) - V_F$ is a non-empty subset of $D - (L_G \cup D')$. This proves our claim. Consequently, $D - (L_G \cup D')$ dominates I' whereas I is a certified dominating set of $F - I'$. This implies that $D - (D' - I)$ is a dominating set of G . In addition, it is no problem to observe that $D - ((D' - I) \cup L')$ is a dominating set of G for every subset L' of L_G (as every $x \in L_G$ is dominated by its only neighbor in $N_G(L_G)$ and $N_G(L_G) \subseteq D$).

Let us consider the function $s: L_G \rightarrow S_G$, where $s(x)$ is the only neighbor of a leaf x , i.e. $s(x)$ is the only element of the set $N_G(x)$. This function is not necessarily an injection, but, since D is a minimal certified dominating set of G , the restriction of s to $L_G \cap D$ is indeed an injection and, in addition, the set $N_G[s(x)]$ is a subset of D for every $x \in L_G \cap D$ (since otherwise, as both x and $s(x)$ belong to D , the set $D - \{x\}$ would be a smaller certified dominating set, a contradiction with the minimality of D). Moreover, the mapping $s: L_G \cap D \rightarrow S_G$ is a bijection between $L_G \cap D$ and $\{y \in S_G: N_G[y] \subseteq D\}$ ($= N_G(L_G \cap D)$). Let S_0 and S_1 be the sets $\{x \in N_G(L_G \cap D): N_G(x) \cap (D' - I) = \emptyset\}$ and $\{x \in N_G(L_G \cap D): N_G(x) \cap (D' - I) \neq \emptyset\}$, respectively. Now let $L_0 = N_G(S_0) \cap L_G$ ($= s^{-1}(S_0)$) and $L_1 = N_G(S_1) \cap L_G$ ($= s^{-1}(S_1)$). Observe that $S_0 \cap S_1 = \emptyset$ and $L_0 \cap L_1 = \emptyset$, whereas $S_0 \cup S_1 = N_G(L_G \cap D) \subseteq S_G^1$ and $L_0 \cup L_1 = L_G \cap D$.

We now prove that the set $D'' = D - ((D' - I) \cup L_1)$ is a certified dominating set of G . Since D'' is a dominating set of G , it suffices to show that no vertex belonging to D'' has exactly one neighbor in $V_G - D''$. To show this, let us take a vertex $x \in D''$. Since $D'' \subseteq D$ and D is a certified dominating set of G , we have $|N_G(x) \cap (V_G - D)| \geq 2$ or $|N_G(x) \cap (V_G - D)| = 0$. In the first case $|N_G(x) \cap (V_G - D'')| \geq 2$, as $V_G - D \subseteq V_G - D''$. Thus assume that $|N_G(x) \cap (V_G - D)| = 0$. Then $N_G[x] \subseteq D$ and, since $x \notin (D' - I) \cup L_1$, x is an element of the set $I \cup S_1 \cup (L_0 \cup S_0)$. If $x \in I$, then $|N_F(x) \cap (D' - I)| \geq 2$ (by the definition of I) and therefore $|N_G(x) \cap (V_G - D'')| \geq 2$ (as $N_F(x) \subseteq N_G(x)$ and $D' - I \subseteq V_G - D''$). If $x \in S_1$ ($= N_G(L_1)$), then $N_G(x) \cap L_1 \neq \emptyset$ and $N_G(x) \cap (D' - I) \neq \emptyset$, and therefore $|N_G(x) \cap (V_G - D'')| \geq 2$ as L_1 and $D' - I$ are disjoint subsets of $V_G - D''$. It remains to show that $N_G(x) \cap (V_G - D'') = \emptyset$ (or, equivalently, that $N_G(x) \subseteq D''$) if $x \in L_0 \cup S_0$. Since $D'' = D - ((D' - I) \cup L_1)$, it suffices to show that $N_G(x) \subseteq D$, $N_G(x) \cap (D' - I) = \emptyset$ and $N_G(x) \cap L_1 = \emptyset$ if $x \in L_0 \cup S_0$. We know already that $N_G[S_0] \subseteq D$, and hence

$$N_G(L_0 \cup S_0) = N_G(L_0) \cup N_G(S_0) = S_0 \cup N_G(S_0) = N_G[S_0] \subseteq D,$$

which proves that $N_G(x) \subseteq D$ if $x \in L_0 \cup S_0$. It follows from the definition of the set S_0 that $N_G(S_0) \cap (D' - I) = \emptyset$ and therefore $N_G[S_0] \cap (D' - I) = \emptyset$ (as S_0 and D' are disjoint). Consequently $N_G(L_0 \cup S_0) \cap (D' - I) = N_G[S_0] \cap (D' - I) = \emptyset$ and this proves that $N_G(x) \cap (D' - I) = \emptyset$ if $x \in L_0 \cup S_0$. Finally, it follows from the properties of the sets S_0 , S_1 , L_0 and L_1 that $N_G(L_1) \cap (L_0 \cup S_0) = S_1 \cap (L_0 \cup S_0) = \emptyset$ and consequently $N_G(x) \cap L_1 = \emptyset$ if $x \in L_0 \cup S_0$. This completes the proof of the fact that D'' is a certified dominating set of G , which, however, contradicts the minimality of

D (as D'' is a proper subset of D). This proves that the set $\{v \in D: N_G[v] \subseteq D\}$ is a subset of $L_G \cup S_G^1$, and therefore the induced subgraph $G[\{v \in D: N_G[v] \subseteq D\}]$ is a corona. $\square$

Now we give a characterization of the graphs for which the upper certified domination number is equal to their order.

Theorem 3.2. *Let G be a graph of order n . Then the following statements are equivalent:*

- (1) *every non-trivial component of G is a corona;*
- (2) $\gamma_{\text{cer}}(G) = n$;
- (3) $\Gamma_{\text{cer}}(G) = n$.

Proof. The equivalence of (1) and (2) has been proved in [2]. It remains to prove the equivalence of (2) and (3). If $\gamma_{\text{cer}}(G) = n$, then $n = \gamma_{\text{cer}}(G) \leq \Gamma_{\text{cer}}(G) \leq n$ and, therefore, $\Gamma_{\text{cer}}(G) = n$. Assume now that $\Gamma_{\text{cer}}(G) = n$. Then V_G is a minimal certified dominating set of G , and, consequently, no proper subset of V_G is a certified dominating set of G . This implies that it cannot be $\gamma_{\text{cer}}(G) < n$, and, thus, it must be $\gamma_{\text{cer}}(G) = n$. $\square$

It follows from the definition of a certified dominating set that if G is a graph of order n , then no $n - 1$ vertices form a certified dominating set of G . Consequently, either $\Gamma_{\text{cer}}(G) = n$ or $\Gamma_{\text{cer}}(G) \leq n - 2$. It is natural then to characterize all graphs G of order n for which $\Gamma_{\text{cer}}(G) = n - 2$. We need the following definitions and notation. A *simple diadem* is a graph obtained from a corona by adding one new vertex and joining it to exactly one support vertex of the corona, whereas a *diadem* is a graph obtained from a corona by adding one new vertex and joining it to one leaf and its neighbor in the corona. Finally, by $G + H$ we denote the join of graphs G and H .

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$. Then $\Gamma_{\text{cer}}(G) = n - 2$ if and only if G is a simple diadem, a diadem, or one of the graphs $K_2 + \overline{K}_{n-2}$ and $\overline{K}_2 + \overline{K}_{n-2}$.*

Proof. It is a simple matter to observe that if a connected graph G of order $n \geq 3$ is a simple diadem, a diadem, $K_2 + \overline{K}_{n-2}$ or $\overline{K}_2 + \overline{K}_{n-2}$, then $\Gamma_{\text{cer}}(G) = n - 2$.

Assume now that G is a connected graph of order $n \geq 3$ such that $\Gamma_{\text{cer}}(G) = n - 2$. Let D be a Γ_{cer} -set of G , and let x and y be the only vertices in $V_G - D$. Let D_0 and D_2 be the sets $\{v \in D: N_G[v] \subseteq D\}$ and $\{v \in D: \{x, y\} \subseteq N_G(v)\}$, respectively.

Assume first that $D_0 = \emptyset$. Then $D_2 = D = V_G - \{x, y\}$ and the minimality of D easily implies the independence of D . Thus $G[D] = \overline{K}_{n-2}$ and now, from the fact that every vertex in D is adjacent to both x and y , it follows that G is one of the graphs $K_2 + \overline{K}_{n-2}$ and $\overline{K}_2 + \overline{K}_{n-2}$ (depending on whether or not x and y are adjacent).

Assume now that $D_0 \neq \emptyset$. It follows from Lemma 3.1 that every vertex in D_0 is a leaf or a weak support vertex of G and the induced subgraph $G[D_0]$ of G is a corona. We consider two subcases $|D_2| = 1$ and $|D_2| \geq 2$ separately.

First assume that $|D_2| = 1$ and let z be the only vertex in D_2 . Let F be a connected component of $G[D_0]$. It follows from the connectivity of G that z is adjacent to at

least one support vertex of F if F is of order at least four. If F is of order two, then z is adjacent to exactly one vertex of F (for otherwise z would be adjacent to both vertices of F and then the proper subset $D - V_F$ of D would be a certified dominating set of G , contrary to the minimality of D). From the above and from the fact that z is adjacent to both x and y it follows that G is a diadem or a simple diadem (depending on whether or not x and y are adjacent).

Finally assume that $|D_2| \geq 2$. We shall prove that this case is impossible. The connectivity of G implies that there exists a vertex $u \in D_2$ adjacent to a support vertex belonging to D_0 . Let us consider the sets $L = \{t \in L_G : d_G(t, u) = 2\}$ and $D' = D - (\{u\} \cup L)$. Now we claim that the set D' is a certified dominating set of G . It is obvious that $L \subseteq L_G \subseteq D_0$ and therefore $D' = (D_2 - \{u\}) \cup (D_0 - L)$ and $V_G - D' = \{x, y, u\} \cup L$. The vertices x and y are dominated by every vertex belonging to the non-empty set $D_2 - \{u\}$. If $t \in L$, then its only neighbor is in $D_0 - L$ and it dominates both t and u . This proves that D' is a dominating set of G . Thus it remains to observe that no vertex belonging to D' has exactly one neighbor in $V_G - D'$. This is obvious for every vertex t in $D_2 - \{u\}$, since x and y are two neighbors of t in $V_G - D'$. Thus assume that t is in $D_0 - L$. Then, since

$$D_0 - L = (S_G \cup L_G) - L = N_G(L) \cup N_G[L_G - L],$$

either $t \in N_G(L)$ or $t \in N_G[L_G - L] = (L_G - L) \cup N_G(L_G - L)$. If $t \in N_G(L)$, then u and the only vertex in $N_G(t) \cap L$ are two neighbors of t in $V_G - D'$. If $t \in L_G - L$, then the only element of $N_G(t)$ belongs to $N_G(L_G - L)$ and so $N_G(t) \cap (V_G - D') = \emptyset$ (since $N_G(L_G - L) \subseteq N_G[L_G - L] \subseteq D'$). Finally, if $t \in N_G(L_G - L)$, then $t \in N_G(L_G) - N_G(L)$ and $N_G(t) \cap (V_G - D') = \emptyset$ (because $V_G - D' = \{x, y, u\} \cup L$, $N_G(\{x, y, u\} \cup L) \subseteq \{x, y\} \cup D_2 \cup N_G(L)$, and $t \notin \{x, y\} \cup D_2 \cup N_G(L)$). This proves that the proper subset $D' = D - (\{u\} \cup L)$ of D is a certified dominating set of G , contrary to the minimality of D . Therefore the case $|D_2| \geq 2$ is impossible, which completes the proof. $\square$

Next, we study the relation between upper domination number and upper certified domination number of a graph. The equality $\gamma(G) = \gamma_{\text{cer}}(G)$ for any graph G with $\delta(G) \geq 2$ (see Corollary 2.3) could suggest that the analogous equality holds for the parameters Γ and Γ_{cer} . Graph G of Figure 3 shows that this is not the case.

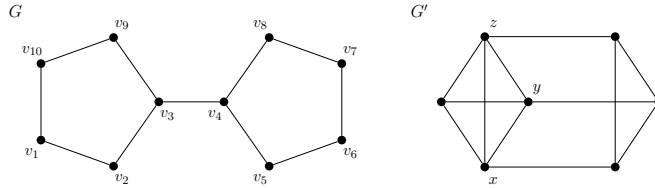


Fig. 3. Graphs G and G'

For this graph we have $\delta(G) = 2$, $\Gamma(G) = 5$ and $\Gamma_{\text{cer}}(G) = 4$, and $\{v_2, v_3, v_6, v_7, v_9\}$ and $\{v_2, v_4, v_6, v_{10}\}$ are examples of Γ - and Γ_{cer} -sets of G , respectively. For graphs G with $\delta(G) \geq 2$, in fact, we always have $\Gamma_{\text{cer}}(G) \leq \Gamma(G)$.

Lemma 3.4. *If G is a graph with $\delta(G) \geq 2$, then $\Gamma_{\text{cer}}(G) \leq \Gamma(G)$.*

Proof. Let D be a Γ_{cer} -set of G . Since $\delta(G) \geq 2$, it follows from Lemma 3.1 that the set $\{v \in V_G : N_G[v] \subseteq D\}$ is empty. Hence D is also a minimal dominating set of G which implies that $\Gamma_{\text{cer}}(G) \leq \Gamma(G)$. $\square$

Taking into account the above lemma, it is natural then to characterize all graphs G with $\delta(G) \geq 2$ for which $\Gamma(G) = \Gamma_{\text{cer}}(G)$. Here we have the following theorem.

Theorem 3.5. *Let G be a connected graph with $\delta(G) \geq 2$. If G has an independent Γ -set, then $\Gamma(G) = \Gamma_{\text{cer}}(G)$.*

Proof. Let D be an independent Γ -set of G . If $v \in D$, then $N_G(v) \subseteq V_G - D$ (as D is independent), $|N_G(v)| \geq 2$ (as $\delta(G) \geq 2$), and therefore $|N_G(v) \cap (V_G - D)| \geq 2$. This proves that D is a minimal certified dominating set of G and implies the inequality $\Gamma(G) \leq \Gamma_{\text{cer}}(G)$. The last inequality and Lemma 3.4 yield the equality $\Gamma(G) = \Gamma_{\text{cer}}(G)$. $\square$

As we have just seen, for graphs G with $\delta(G) \geq 2$ the equality $\Gamma(G) = \Gamma_{\text{cer}}(G)$ holds, if G has an independent Γ -set. It should be noted that the converse implication, however, is not true. For example, one can check that for the graph G' shown in Figure 3 is $\delta(G') \geq 2$ and $\Gamma(G') = \Gamma_{\text{cer}}(G') = 3$, but the only Γ -set $\{x, y, z\}$ of G' is not independent.

The *independence number* $\beta_0(G)$ of a graph G is the cardinality of a largest independent set of vertices of G . It is well-known that $\beta_0(G) \leq \Gamma(G)$ for every graph G . Therefore, by Theorem 3.5, we immediately have our final corollary.

Corollary 3.6. *If G is a graph with $\delta(G) \geq 2$ and $\beta_0(G) = \Gamma(G)$, then also $\beta_0(G) = \Gamma(G) = \Gamma_{\text{cer}}(G)$.*

The equality of the parameters $\beta_0(G)$ and $\Gamma(G)$ has been studied by a number of authors (see for instance [1] and [7, pp. 80–84], and the references there) for well-known families of graphs, including strongly perfect graphs and their different subclasses: bipartite graphs, chordal graphs, and circular arc graphs, just to name a few. For all such graphs G , the equality $\Gamma(G) = \Gamma_{\text{cer}}(G)$ is true if $\delta(G) \geq 2$.

4. CLOSING OPEN PROBLEMS

We close with the following list of open problems that we have yet to settle.

Problem 4.1. *Determine the class of graphs G for which $\gamma_{\text{cer}}(G) = \Gamma_{\text{cer}}(G)$.*

Problem 4.2. *Determine all the trees T for which $\gamma_{\text{cer}}(T) = \gamma(T)$.*

Problem 4.3. Let a, b, c, d be positive integers with $a \leq b \leq c \leq d$. Find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\Gamma(G) = b$, $\gamma_{\text{cer}}(G) = c$ and $\Gamma_{\text{cer}}(G) = d$. Similarly, find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\gamma_{\text{cer}}(G) = b$, $\Gamma(G) = c$ and $\Gamma_{\text{cer}}(G) = d$. Finally, find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\gamma_{\text{cer}}(G) = b$, $\Gamma_{\text{cer}}(G) = c$ and $\Gamma(G) = d$.

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Certified domination critical graphs upon vertex removal

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ABSTRACT

A set D of vertices of a graph $G = (V_G, E_G)$ is a *dominating set* of G if every vertex in $V_G - D$ is adjacent to at least one vertex in D . The *domination number* of a graph G , denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G . A subset $D \subseteq V_G$ is called a *certified dominating set* of G if D is a dominating set of G , and every vertex in D has either zero or at least two neighbours in $V_G - D$. The cardinality of a smallest certified dominating set of G is called the *certified domination number* of G , and it is denoted by $\gamma_{\text{cer}}(G)$. A vertex v of G is *certified critical* if $\gamma_{\text{cer}}(G - v) < \gamma_{\text{cer}}(G)$, and a graph G is *vertex certified domination critical* or γ_{cer} -critical if the removal of any vertex reduces its certified domination number. In this paper, we give examples and properties of certified critical vertices and vertex certified domination critical graphs. As an example of an application of certified critical vertices, we give a constructive characterisation of trees for which the smaller partite set is a minimum certified dominating set.

Keywords: certified domination, certified domination critical graphs, vertex-critical graphs, constructive characterization, bipartite graphs, trees, graph factorization, domination theory

2020 Mathematics Subject Classification: 05C69, 05C70, 05C05.

1. Introduction

All graphs discussed here are finite and undirected with no loops and multiple edges. We generally follow the notation and terminology of [11]. For a graph G , V_G denotes the

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set of vertices, while E_G denotes the edge set. The number of vertices in a graph G is called the *order* of G , denoted by $n = |V_G|$. In a connected graph G , the *distance* $d_G(u, v)$ from a vertex u to a vertex v is the smallest length of a u - v path in G . The *diameter* $\text{diam}(G)$ of a connected graph G is the greatest distance between any two vertices of G . For a vertex $v \in V_G$, the *open neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v , and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v . The *open neighbourhood* of a set $X \subseteq V_G$ is $N_G(X) = \bigcup_{v \in X} N_G(v)$, while the *closed neighbourhood* of X is $N_G[X] = N_G(X) \cup X$. The set of its leaves (vertices of degree one) is denoted by L_G , the set of support vertices (neighbors of leaves) is denoted by S_G , the subset of weak support vertices adjacent to exactly one leaf is denoted by S_G^1 , while the corona of a graph is constructed by adding a unique pendant vertex to each of its original vertices.

Given a graph G , we say that a subset $D \subseteq V_G$ is a *dominating set* of G if every vertex in $V_G - D$ has a neighbour in D , while D is *2-dominating* if every vertex in $V_G - D$ has at least two neighbours in D . The *domination number* of a graph G , denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G . A dominating set of G of minimum cardinality is called a γ -set of G . A subset $D \subseteq V_G$ is called a *certified dominating set* of G if D is a dominating set of G and every vertex belonging to D has either zero or at least two neighbours in $V_G - D$. The cardinality of a smallest certified dominating set of G is called the *certified domination number* of G , and is denoted by $\gamma_{\text{cer}}(G)$. A certified dominating set of G of minimum cardinality is called a γ_{cer} -set of G . For later reference, we recall without proof several further standard variants: a *total dominating set* requires every vertex to be adjacent to a vertex in the set, and its minimum size is the *total domination number* $\gamma_t(G)$; a *paired dominating set* induces a perfect matching, and its minimum size is $\gamma_p(G)$; an *independent dominating set* is dominating and independent, with minimum size $\gamma_i(G)$; a *Roman dominating function* $f: V_G \rightarrow \{0, 1, 2\}$ assigns 2 to at least one neighbour of each vertex with $f(v) = 0$, and the minimum weight $\sum_v f(v)$ is the *Roman domination number* $\gamma_R(G)$; finally a *power dominating set* is a set whose iterative forcing process observes all vertices, and the minimum size of such a set is $\gamma_P(G)$. Whenever ψ denotes any of these parameters, a vertex v is ψ -critical if $\psi(G - v) < \psi(G)$.

Certified domination was introduced by Dettlaff et al. [8] to model certain social network relations. In [21], sufficient conditions were given for the equality of the domination number $\gamma(G)$ and the certified domination number $\gamma_{\text{cer}}(G)$ of a graph G . Moreover, relationships between certified dominating sets and 2-dominating sets were established in [22, 23].

Recently, numerous works have advanced the study of certified domination from structural, algorithmic, and applied perspectives. For instance, Jakkepalli et al. [19] and Raczek and Miotk [25] proved that determining $\gamma_{\text{cer}}(G)$ is NP-complete in general (even for various restricted graph classes) and presented efficient algorithms for computing $\gamma_{\text{cer}}(G)$ in special cases. In addition, new variants of the concept have been introduced. Hamja [12] defined the *certified perfect dominating set*, which integrates the perfect domination condition into certified domination. Meanwhile, the *connected certified domination number*, which requires a dominating set that is both certified and whose induced subgraph is connected, was first investigated in [2] and later characterized for critical cases by Lone

and Goswami [1].

Several recent studies have also examined structural properties of $\gamma_{\text{cer}}(G)$ and its applications. Pushpam et al. [24] analyzed the stability of the certified domination number under edge addition, identifying families of graphs in which $\gamma_{\text{cer}}(G)$ remains invariant when new edges are added. Durai Raj and Kumari [26] investigated $\gamma_{\text{cer}}(G)$ in graph products, determining exact values for certain graph constructions. Notably, certified domination has been applied in network design and optimization. Raczek and Miotk [18] utilized $\gamma_{\text{cer}}(G)$ to model and optimize fire-fighting water supply networks, demonstrating the effectiveness of certified dominating sets in meeting resource constraints. The same authors subsequently proposed two heuristic approaches for selecting influential nodes in social networks using certified dominating sets [25], underscoring the practical relevance of $\gamma_{\text{cer}}(G)$ in real-world scenarios.

Brigham, Chinn, and Dutton [5] began the study of vertex domination-critical graphs by investigating how the domination number decreases upon the removal of any single vertex. Further properties of these graphs were explored in [7, 10, 9, 31, 30]. However, a complete characterisation of domination-critical graphs is still unknown. Various types of critical graphs have been examined with respect to domination parameters (considering vertex or edge removal, as well as vertex or edge addition), including paired-domination-critical, double-domination-critical, and isolate-domination-critical cases in [17, 27, 28]. In this paper, we are interested in studying the effect that graph modifications have on the certified domination number. More precisely, we first investigate graphs for which the certified domination number decreases following the removal of any vertex. We call such a vertex *certified critical*. (In [8], the effect on the certified domination number of adding a vertex or an edge was previously examined.) As an example of the utility of certified critical vertices, we provide a sufficient condition in bipartite graphs under which the smaller partite set is a minimum certified dominating set. For trees, we give a constructive, complete characterisation of those graphs whose smaller partite set is a γ_{cer} -set. Finally, we provide a characterisation of 2-vertex certified domination-critical graphs using the terminology of the graph factorisation problem.

2. Application of critical vertices in domination variants

An *influence vertex* of a domination parameter $\psi(G)$ is a vertex whose removal necessitates a qualitative adjustment in the optimal size of an existing dominating set under such a model. The identification of these vertices reveals *where* a network is structurally vulnerable and *how* its resource needs are affected by a localized disruption.

Standard domination (γ). In the classical environment, a γ -critical vertex reveals hidden redundancy if $\gamma(G-v) < \gamma(G)$ or identifies a point of unique failure when $\gamma(G-v) > \gamma(G)$ [5]. Freight-logistics hubs whose loss raises γ mark warehouses whose protection avoids costly re-routing.

Total domination (γ_t). As every vertex must be backed up by a neighbour, γ_t -critical vertices highlight sensitivity in *local backup* systems such as peer-to-peer monitors or redundant servers [6].

Paired domination (γ_p). Here dominating vertices appear in mutually-aiding pairs. A γ_p -critical vertex is on a bottleneck matching edge: its removal destroys a minimal pair and compels the network to hire an *entire extra pair* of resources [14]. Guard patrols or twin-server clusters are natural analogies.

Independent domination (γ_i). While dominating vertices do not have to be non-adjacent, γ_i -critical nodes provide the interference versus coverage trade-off. For wireless sensor grids a critical access point can result in extra nodes being deployed (if γ_i is increased) or allow a sparse, interference-free network (if γ_i is reduced) [3].

Power domination (γ_P). A power-critical bus in an electric network is one whose loss requires extra Phasor Measurement Units in order to restore full observability [15]. The grid operators therefore view such buses as topological "linchpins" for state-estimation redundancy.

Let a regional distribution network be represented by a bipartite graph $G = ((A, B); E_G)$, with the set A representing possible distribution centers and the set B representing retail outlets.

A *certified dominating set* $D_{\text{cer}} \subseteq A$ is related to the opened centres: every outlet in B is within reach of at least one opened hub, and every hub serves either no outlet (unused back-up location) or at least two outlets (economies of scale). This "0 or ≥ 2 " requirement precisely models a common cost constraint: it is uneconomical to operate a centre to serve a single outlet, but demanding two centres per outlet (as in double domination) is too costly. Let G be the star $K_{1,3}$ with central vertex $v \in A$ adjacent to three outlets $b_1, b_2, b_3 \in B$. As v dominates every outlet and satisfies the certified condition ($|N_G(v) \cap B| = 3 \geq 2$), we obtain $\gamma_{\text{cer}}(G) = 1$ with $D_{\text{cer}} = \{v\}$. Vertex v is therefore *certified-domination-critical*: removing it isolates the outlets, and each outlet becomes a micro-hub in itself, yielding the certified dominating set $D'_{\text{cer}} = \{b_1, b_2, b_3\}$ and raising the parameter to $\gamma_{\text{cer}}(G - v) = 3$. In terms of operations, the removal of v triples the number of active facilities, ballooning fixed costs and complicating inventory flows. Therefore certified domination-critical vertices identify the hubs whose presence sustains the optimal balance between efficiency (minimal centres) and redundancy (no single-client centres). Finding them in larger supply graphs signals precisely where to maximize contingency stock or redundant routing capacity, providing practical guidance for resilient cost-effective logistics planning.

3. Vertex certified domination critical graph

Let $G = (V_G, E_G)$ be a graph. A vertex v is *certified critical* if $\gamma_{\text{cer}}(G - v) < \gamma_{\text{cer}}(G)$. Additionally, G is *vertex certified domination critical*, or γ_{cer} -critical, if each vertex of G is certified critical.

In [8], it was observed that, for a graph G of order n , $1 \leq \gamma_{\text{cer}}(G) \leq n$, it is obvious that, if $\gamma_{\text{cer}}(G) = 1$, then G is vertex certified domination critical graph if and only if $G = K_1$. The next question is to characterise a vertex certified domination critical graph G , for which $\gamma_{\text{cer}}(G) = n$. This gives rise to the following Observation.

Observation 3.1. *Let G be a graph of order n . If $\gamma_{\text{cer}}(G) = n$, then G is a vertex certified domination critical graph.*

Proof. Suppose that $\gamma_{\text{cer}}(G) = n$ and G is not a γ_{cer} -critical graph. Then there exists a vertex v of G such that $\gamma_{\text{cer}}(G - v) \geq \gamma_{\text{cer}}(G) = n$. Hence, v must belong to every γ_{cer} -set of G , which contradicts the fact that $\gamma_{\text{cer}}(G - v) = n$. $\square$

In [8], it was demonstrated that, for a graph G of order n , $\gamma_{\text{cer}}(G) = n$ if and only if G is the complement of a complete graph, the corona of a graph, or the union of both. Taking into account the above result and Observation 3.1, we have the following Corollary.

Corollary 3.2. *Let G be a graph of order n . If G is the complement of a complete graph, the corona of a graph, or the union of both, then G is a vertex certified domination critical graph.*

Figure 1 presents graphs G and H , which are γ_{cer} -critical and not γ_{cer} -critical, respectively. Notice that the graph G is the corona, which means that the vertices of G must belong to the minimum certified dominating set. It follows from Corollary 3.2 that G is a vertex certified domination critical graph. For a graph H , the unique minimum certified dominating set is $\{u_1, u_3\}$, and it is easy to check that, for every vertex of H , the certified domination number does not decrease.

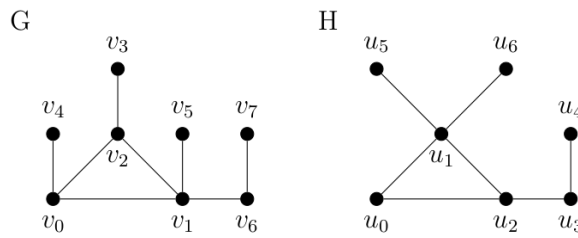


Fig. 1. Graphs G and H

Now, we focus on characterising the most common classes of graphs, namely vertex certified domination critical graphs. In [8], it was observed that, for a complete graph K_n of order n , we have $\gamma_{\text{cer}}(K_n) = 1$ if $n \neq 2$ or $\gamma_{\text{cer}}(K_n) = 2$ if $n = 2$. For a cycle C_n of order n , the certified domination number $\gamma_{\text{cer}}(C_n) = \lceil n/3 \rceil$ if $n \geq 3$. For a path P_n of order n , we have $\gamma_{\text{cer}}(P_1) = 1$, $\gamma_{\text{cer}}(P_2) = 2$, $\gamma_{\text{cer}}(P_4) = 4$, or $\gamma_{\text{cer}}(P_n) = \lceil n/3 \rceil$ if $n = 3$ or $n \geq 5$. Additionally, it was also proven that for a graph G , every support vertex must belong to every γ_{cer} -set of G . In the observations given below, a characterisation is presented of vertex certified domination critical complete graphs and cycles.

Observation 3.3. *Let G be a complete graph of order n . Then G is a vertex certified domination critical graph if and only if $n \in \{1, 2\}$.*

Proof. Suppose that the complete graph G is γ_{cer} -critical, and let $n \geq 3$. Then $\gamma_{\text{cer}}(G) =$

1. Additionally, since the graph $G' = G - v$ for some vertex v of G , is a complete graph of order $n - 1$, we have the contradiction $\gamma_{\text{cer}}(G') \geq \gamma_{\text{cer}}(G)$.

On the other hand, if $n = 1$ and v is a vertex of G , then $\gamma_{\text{cer}}(G) = 1$ and $\gamma_{\text{cer}}(G') = 0$, where $G' = G - v$. If $n = 2$, then G is a corona graph of K_1 , and from Corollary 3.2, it follows that G is a γ_{cer} -critical graph. $\square$

Observation 3.4. *Let G be a cycle of order n , where $n \geq 3$. Then G is a vertex certified domination critical graph if and only if $n = 3k + 1$ for $k \in \mathbb{N}^+$.*

Proof. Let G be a cycle of order n , where $n \geq 3$ and G is γ_{cer} -critical. Let $n \neq 3k + 1$ for $k \in \mathbb{N}^+$. We need to consider the following two cases. If $n = 3k$, then $\gamma_{\text{cer}}(G) = k$. Let v be a vertex of G . If $k = 1$, then $G' = G - v = K_2$ and $\gamma_{\text{cer}}(G') = 2 > \gamma_{\text{cer}}(G)$, which is a contradiction, namely that G is γ_{cer} -critical. Now, assume that $k > 1$. Hence, the graph $G' = G - v$ is the path of order $3k - 1$ and $\gamma_{\text{cer}}(G') = k = \gamma_{\text{cer}}(G)$, which is another contradiction, namely that G is a γ_{cer} -critical graph. Now, suppose that $n = 3k + 2$. Then $\gamma_{\text{cer}}(G) = k + 1$. Let v be a vertex of G . Then the graph $G' = G - v$ is the path of order $3k + 1$, and we have $\gamma_{\text{cer}}(G') = k + 1$. This is also a contradiction, namely that G is γ_{cer} -critical graph.

Suppose that G is a cycle of order $3k + 1$ for $k \in \mathbb{N}^+$. Suppose that G is not a γ_{cer} -critical graph. Then there exists a vertex v of G such that $\gamma_{\text{cer}}(G - v) \geq \gamma_{\text{cer}}(G)$. The certified domination number of G is equal to $k + 1$, while $\gamma_{\text{cer}}(G - v) = k$. This contradicts the fact that G is not a γ_{cer} -critical graph, thereby completing the proof. $\square$

The next results show that there does not exist a certified critical strong support vertex. We start with the following observation.

Observation 3.5. *Let $G = (V_G, E_G)$ be a graph. If there exists a strong support vertex s in G , then there exists a vertex $u \in V_G$ such that $\gamma_{\text{cer}}(G - u) \geq \gamma_{\text{cer}}(G)$.*

Proof. Let s be a strong support vertex of G and let $L = N_G(s) \cap L_G$ be a set of leaves of s . Now, let $u \in L$. We have $\gamma_{\text{cer}}(G - u) \geq \gamma_{\text{cer}}(G)$, since every support vertex belongs to every minimum certified dominating set. $\square$

Based the above observation, we obtain the following Corollary.

Corollary 3.6. *If G is a vertex certified domination critical graph, then there is no strong support vertex in the graph G .*

Our next goal is to characterize the certified critical leaves and support vertices using the private neighbourhood notation and shadowed vertex definition. Recall that the *private neighbourhood* $pn[v, S]$ of $v \in S$ is defined by $pn[v, S] = N[v] - N[S - \{v\}]$ for $S \subseteq V_G$ of the graph G . Equivalently, $pn[v, S] = \{u \in V : N[u] \cap S = \{v\}\}$. Each vertex in $pn[v, S]$ is called a *private neighbour* of v . A vertex v of a graph G is said to be *shadowed* with respect to a certified dominating set D_{cer} if $N_G[v] \subseteq D_{\text{cer}}$, while v is *half-shadowed* if

$|N_G(v) \cap D_{\text{cer}}| = 1$. Consider the graph G from Figure 1, in which all vertices belong to the minimum certified dominating set D_{cer} . Thus, every vertex v of G is shadowed, and it is easy to check that $pn[v, D_{\text{cer}}] = \emptyset$. For the graph H in Figure 1, the unique γ_{cer} -set $D_{\text{cer}} = \{u_1, u_3\}$, and we have $pn[u_1, D_{\text{cer}}] = \{u_0, u_1, u_5, u_6\}$ and $pn[u_3, D_{\text{cer}}] = \{u_3, u_4\}$.

The next result is a characterisation of shadowed vertices, taking into account the private neighbourhood notation.

Lemma 3.7. *Let G be a graph. Then a vertex v is shadowed if and only if $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set D_{cer} of G containing v .*

Proof. Suppose that v is shadowed vertex. Since $N_G[v] \subseteq D_{\text{cer}}$ for some γ_{cer} -set of G , it is clear that $pn[v, D_{\text{cer}}] = \emptyset$.

Now, let $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set of G containing v . Suppose to the contrary that v is not shadowed. Since $pn[v, D_{\text{cer}}] = \emptyset$ and v is not shadowed, we conclude that v is not a support or a leaf (otherwise $pn[v, D_{\text{cer}}] \neq \emptyset$ or v is a half-shadowed vertex, which is impossible in D_{cer}). Therefore, the set $D'_{\text{cer}} = D_{\text{cer}} - \{v\}$ would be a smaller certified dominating set, which yields a contradiction. This completes the proof. $\square$

Lemma 3.8. *Let G be a graph. Then for a vertex v and γ_{cer} -set D_{cer} of G , we have $pn[v, D_{\text{cer}}] = \emptyset$ if and only if v is shadowed and $v \in L_G \cup S_G^1$.*

Proof. Let $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set D_{cer} of G containing v . By Lemma 3.7, it follows that v is a shadowed vertex. Now, it remains to be proven that v is a leaf or weak support vertex. On the contrary, suppose that this is not true. Since $pn[v, D_{\text{cer}}] = \emptyset$, it follows that v is not a strong support vertex and $d_G(v) \geq 2$. Based on the assumption that v is shadowed, then $N_G(v) \subseteq D_{\text{cer}}$, and the set $D'_{\text{cer}} = D_{\text{cer}} - \{v\}$ would be a smaller certified dominating set, thereby yielding a contradiction.

Now, assume that v is shadowed and v is a weak support or a leaf. From Lemma 3.7, we have $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set D_{cer} of G containing v . $\square$

The next results provide the characterisation of certified critical support vertices and leaves.

Lemma 3.9. *Let G be a graph. If v is shadowed vertex of G , then v is certified critical.*

Proof. Let v be a shadowed vertex for some γ_{cer} -set D_{cer} of G containing v . From Lemma 3.7, it follows that $pn[v, D_{\text{cer}}] = \emptyset$. Since v does not contain a private neighbour in D_{cer} , then the set $D'_{\text{cer}} = D_{\text{cer}} - \{v\}$ is certified dominating set in $G - v$. Hence, we obtain $\gamma_{\text{cer}}(G - v) < \gamma_{\text{cer}}(G)$, and therefore v is certified critical. $\square$

Lemma 3.10. *Let G be a graph. If for a weak support s and its leaf l it follows that $pn[s, D_{\text{cer}}] = \{l\}$ for some γ_{cer} -set D_{cer} of G containing s , then l is certified critical.*

Proof. Suppose that $pn[s, D_{\text{cer}}] = \{l\}$ for some γ_{cer} -set D_{cer} of G containing s . Since l is the only private neighbour of s and for every non-leaf neighbour u of s , $N_G(u) \cap (D_{\text{cer}} - \{s\}) \neq \emptyset$, then the set $D'_{\text{cer}} = D_{\text{cer}} - \{s\}$ is a certified dominating set in $G - l$. Hence, $\gamma_{\text{cer}}(G - l) < \gamma_{\text{cer}}(G)$, and therefore l is certified critical. $\square$

Lemma 3.11. *Let G be a graph. A support vertex s is certified critical if and only if s is a shadowed vertex.*

Proof. Let s be certified critical vertex. It follows from Corollary 3.6 that s is a weak support. Suppose to the contrary that s is not a shadowed vertex, that is $N_G[s] \subsetneq D_{\text{cer}}$. Let l be a leaf of s . It follows that $l \notin D_{\text{cer}}$, and we have $\gamma_{\text{cer}}(G - s) \geq \gamma_{\text{cer}}(G)$ since l is an isolated vertex in $G - s$ and l belongs to every γ_{cer} -set of $G - s$. This contradicts our assumption that s is certified critical vertex.

Now, assume that s is a shadowed vertex. From Lemma 3.9, we conclude that s is certified critical, which completes the proof. $\square$

In [9], it was demonstrated that a vertex v is critical if and only if $pn[v, D] = \{v\}$ for some γ -set D of G containing v . Using a similar technique, we have the following Lemma.

Lemma 3.12. *Let G be a graph. If $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v , then v is certified critical.*

Proof. The proof is adapted from [9]. Assume that $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v . Since $N_G(v) \cap D_{\text{cer}} = \emptyset$, the set $D_{\text{cer}} - \{v\}$ is certified dominating set in $G - v$. Consequently, v is certified critical. $\square$

Now, we are ready to give a complete characterisation of vertex certified domination critical graphs. We start from the graphs without leaves and isolated vertices.

Theorem 3.13. *Let G be a graph with a minimum degree of at least two. Then G is vertex certified domination critical graph if and only if for each vertex v of G , $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v .*

Proof. Let G be a γ_{cer} -critical graph. Hence, every vertex of G is certified critical. Suppose to the contrary that there exists a vertex v such that $pn[v, D_{\text{cer}}] \neq \{v\}$ for all γ_{cer} -sets D_{cer} of G containing v . Hence, there exists at least one private neighbour distinct of v . Therefore, the set $D_{\text{cer}} - \{v\}$ would not be a certified dominating set of $G - v$, thereby yielding a contradiction.

Now, suppose that, for each vertex v of G , $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v . It follows from Lemma 3.12 that v is certified critical. Therefore, G is a vertex certified domination critical graph. $\square$

Considering Theorem 3.13, the following result can be obtained.

Theorem 3.14. *Let G be a graph. Then G is a vertex certified domination critical graph if and only if for each vertex v of G , $pn[v, D_{\text{cer}}] = \{v\}$ or $pn[v, D_{\text{cer}}] = \emptyset$ for some γ_{cer} -set D_{cer} of G containing v .*

Proof. Suppose that G is a γ_{cer} -critical graph. Hence, every vertex v of G is certified critical. Conversely, suppose that there exists a vertex v such that $pn[v, D_{\text{cer}}] \neq \emptyset$ and $pn[v, D_{\text{cer}}] \neq \{v\}$ for all γ_{cer} -set D_{cer} of G containing v . Hence, v is not a leaf (otherwise v would be a shadowed vertex) or an isolated vertex. We need to consider the following two cases. If v is support, then v is not shadowed, and it follows from Lemma 3.11 that a contradiction arises with the fact that G is γ_{cer} -critical. Now, we may assume that G is a graph with a minimum degree of at least two. Therefore, v is not certified critical by Theorem 3.13. Hence, G is not a γ_{cer} -critical graph, yielding a contradiction with the initial supposition.

Now, suppose that, for every vertex v of G , $pn[v, D_{\text{cer}}] = \emptyset$ or $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set of G containing v . Then, it follows from Lemmas 3.9 and 3.12 that v is certified critical. Therefore, G is a vertex certified domination critical graph. $\square$

As a consequence of Theorem 3.13 and 3.14, the following corollaries and related results can be identified.

Corollary 3.15. *Let G be a connected graph with at least one leaf. Then G is a vertex certified domination critical graph if and only if G is a corona of some graph G' .*

Proof. Let G be a γ_{cer} -critical graph. Hence, according to Theorem 3.14, it follows that, for every vertex v of G , $pn[v, D_{\text{cer}}] = \emptyset$ or $pn[v, D_{\text{cer}}] = \{v\}$ for some γ_{cer} -set D_{cer} of G containing v . From Observation 3.5, it is reasonable to conclude that G does not contain strong supports. Hence, every support vertex is shadowed. Suppose to the contrary that G is not a corona of some graph G' . Then, there exists a weak support s such that $N_G(s) - (L_G \cup S_G) \neq \emptyset$. Let $u \in N_G(s) - (L_G \cup S_G)$. Thus, since s is shadowed, u must belong to some γ_{cer} -set D_{cer} of G . On the other hand, we have $pn[u, D_{\text{cer}}] \neq \{u\}$, and it follows from Theorem 3.14 that u is not certified critical. Therefore, G is not a γ_{cer} -critical graph, which yields a contradiction.

Assume that G is a corona of some graph G' . Then it follows from Observation 3.1 that G is a γ_{cer} -critical graph. This completes the proof. $\square$

Corollary 3.16. *A tree T is a vertex certified domination critical graph if and only if T is a corona of some tree T' .*

Corollary 3.17. *Let G be a path of order n . Then G is a vertex certified domination critical graph if and only if $n \in \{1, 2, 4\}$.*

Corollary 3.18. *Let G be a graph of with a minimum degree of at least two. Then G is a vertex certified domination critical graph if and only if G is a domination critical graph.*

4. Bipartite graphs with the largest certified domination number

In this section, an application of the definition of certified critical vertices is given. We give a complete characterisation of bipartite graphs $G = ((A, B); E_G)$ for which the smaller partite set A is a minimum certified dominating set. Similar research related to the domination number was undertaken by [13, 20, 4], while [22] gave a partial characterisation of these graphs related to the domination number and 2-dominating sets.

We start with the following Lemma, which are proven in [22].

Lemma 4.1. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is the minimal certified dominating set, then A does not contain a shadowed vertex.*

Proof. Assume that the set A is a minimal certified dominating set. On the contrary, suppose that there exists a shadowed vertex u in A . From Lemma 3.8, u is a leaf or weak support, and $N_G(u) \subseteq B$ must belong to the minimal certified dominating set. This contradicts the fact that A is a minimal certified dominating set. $\square$

As a consequence of the above Lemma, we have the following Corollary.

Corollary 4.2. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is a γ_{cer} -set of G , then A does not contain shadowed vertices, and hence $S_G \subseteq A$.*

Now, we give a sufficient condition of bipartite graphs $G = ((A, B); E_G)$ for which $\gamma_{\text{cer}}(G) = |A|$, where $1 \leq |A| \leq |B|$.

Theorem 4.3. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is a γ_{cer} -set of G , then the following conditions hold:*

- (a) *Every support vertex belongs to the set A ;*
- (b) *Every non-support vertex of the set A is certified critical.*

Proof. Assume that the set A is a γ_{cer} -set of G . The condition (a) follows from Corollary 4.2. Suppose to the contrary that there exists a non-support vertex v of A , which is not certified critical. Then $|A| - 1 = \gamma_{\text{cer}}(G - v) \geq \gamma_{\text{cer}}(G) = |A|$, which is impossible. This proves condition (b). $\square$

Let $\mathcal{T}_{\max}$ denote the set of trees for which the smaller partite set A is a minimum certified dominating set. We now provide a constructive characterisation of the trees belonging to the set $\mathcal{T}_{\max}$. Similar constructive characterisation of trees for different domination parameters or properties have been presented in other studies, including [16, 20, 29]. Our characterisation is based on two simple operations, and it is similar to the result obtained in [20].

Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence of trees $T_0, \dots, T_k$, where $k \geq 0$, $T_0 = S_{1,n}$ ($n \geq 2$) and $T = T_k$. In addition, if $k \geq 1$, then for each

$i = 1, \dots, k$, the tree T_i can be obtained from the tree T_{i-1} with the bipartition (A', B') , where $|A'| \leq |B'|$. This relies on one of the following two operations: firstly, $\mathcal{O}_1$; and secondly, $\mathcal{O}_2$. These operations are defined and illustrated in Figure 2.

$\mathcal{O}_1$ Add a new vertex b to T' and join b to an A' -vertex a' of T' ;

$\mathcal{O}_2$ Add two new vertices a and b to T' and join a to b to B' -vertex b' of T' , which is not certified critical in T' .

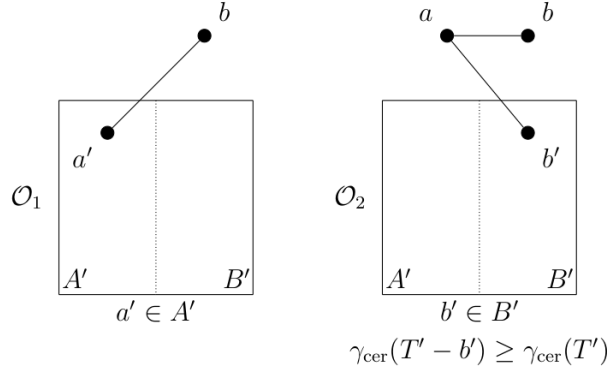


Fig. 2. Operations $\mathcal{O}_1$ and $\mathcal{O}_2$

Each of the vertices a' in the above defined operation $\mathcal{O}_1$ and b' in the operation $\mathcal{O}_2$ are denoted the *attachers* of T' . We shall prove that the trees in $\mathcal{T}$ are precisely the trees belonging to the family $\mathcal{T}_{max}$. Our first aim is to show that each tree in $\mathcal{T}$ belongs to the family $\mathcal{T}_{max}$. For this purpose, we prove the following Lemma.

Lemma 4.4. *If $T \in \mathcal{T}$, then $T \in \mathcal{T}_{max}$.*

Proof. Let $T_0, T_1, T_2, \dots$ be a sequence of trees, where $T_0 = S_{1,k}$ ($k \geq 2$) and every T_n can be obtained from the tree T_{n-1} by one of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$. By induction on n , we shall prove that $T_n \in \mathcal{T}_{max}$ for every $n \in \mathbb{N}$. If $n = 0$, then $T_0 = S_{1,k}$ ($k \geq 2$) and $T_0 \in \mathcal{T}_{max}$ (as $\gamma_{cer}(S_{1,k}) = 1 = |A|$). Suppose that $n \geq 1$ and $T_{n-1} = ((A', B'), E_{T_{n-1}})$ is a tree, where (A', B') is a bipartition of T_{n-1} for which $|A'| \leq |B'|$, and the set A' is a γ_{cer} -set of T_{n-1} . We consider two cases, depending on which operation is used to construct the tree $T = T_n$ from $T' = T_{n-1}$.

Case 1. T is obtained from T' by operation $\mathcal{O}_1$. In this case, $(A', B' \cup \{b\})$ is the bipartition of T and $|A'| < |B' \cup \{b\}|$. Now, since A' is γ_{cer} -set of T' and the attacher a' is support vertex, it follows that the set A' is a γ_{cer} -set of T . This proves that $T \in \mathcal{T}_{max}$.

Case 2. T is obtained from T' by operation $\mathcal{O}_2$. In this case, $(A' \cup \{a\}, B' \cup \{b\})$ is the bipartition of T and $|A' \cup \{a\}| \leq |B' \cup \{b\}|$. Since a is support vertex, we have $\gamma_{cer}(T) \leq |A' \cup \{a\}| \leq |A'| + 1$. We claim that the set $A' \cup \{a\}$ is a γ_{cer} -set of T . Since a is a support vertex and a must belong to every γ_{cer} -set of T , then it remains to be proven that $\gamma_{cer}(T) \geq |A'| + 1$. Let D_{cer} be a γ_{cer} -set of T . Then $\gamma_{cer}(T) = |D_{cer}| \leq |A'| = \gamma_{cer}(T')$. Since a is support vertex, then $a \in D_{cer}$. Now, the attacher b' is dominated only by

a (otherwise $D_{\text{cer}} - \{a\}$ would be a certified dominating set of T' , and we would have $\gamma_{\text{cer}}(T') \leq |D_{\text{cer}} - \{a\}| < |A'| = \gamma_{\text{cer}}(T')$, yielding a contradiction). Then $D_{\text{cer}} - \{a\}$ is certified dominating set of $T' - b'$. Consequently, $\gamma_{\text{cer}}(T' - b') \leq |D_{\text{cer}} - \{a\}| < |A'|$, which contradicts the fact that b' is not a certified critical vertex in T' , that is $\gamma_{\text{cer}}(T' - b') \geq \gamma_{\text{cer}}(T')$ for the attacher b' in operation $\mathcal{O}_2$. This proves that $\mathcal{T} \in \mathcal{T}_{\text{max}}$, completing the proof. $\square$

We are now ready to provide a constructive characterisation of the trees belonging to the family $\mathcal{T}_{\text{max}}$.

Theorem 4.5. $\mathcal{T} = \mathcal{T}_{\text{max}}$.

Proof. It follows from Lemma 4.4 that $\mathcal{T} \subseteq \mathcal{T}_{\text{max}}$. Conversely, suppose T is a tree belonging to the family $\mathcal{T}_{\text{max}}$. Let (A, B) be a bipartition of T , where $1 \leq |A| \leq |B|$. By induction on the order of T , we shall prove that, if the set A is a γ_{cer} -set of T , then T belongs to $\mathcal{T}$, that is T can be obtained from $S_{1,k}$ ($k \geq 2$) (which belongs to $\mathcal{T}$ and $\mathcal{T}_{\text{max}}$) by repeated applications of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Since a star $S_{1,k}$ ($k \geq 2$) belongs to $\mathcal{T}$ and $\mathcal{T}_{\text{max}}$, we may assume that $\text{diam}(T) \geq 3$. By Observation 3.2 and Theorem 4.3, we may assume that $|A| < |B|$ and $|A \cap L_T| = \emptyset$. Let $(x_0, x_1, \dots, x_d)$ be the longest path in T . Since x_0 and x_d are leaves in T and $d = \text{diam}(T) \geq 3$, it is necessarily the case that $x_0 \in B$, $x_1 \in A, \dots, x_d \in B$, and therefore $d \geq 4$. If $d_T(x_1) > 2$, then $(A, B - \{x_0\})$ is a bipartition of the subtree $T' = T - x_0$ of T , and it is clear that the set A is a γ_{cer} -set of T' . Thus, $T' \in \mathcal{T}_{\text{max}}$ and the inductive hypothesis imply that $T' \in \mathcal{T}$. At this point, the tree T can be rebuilt from T' by applying the operation $\mathcal{O}_1$ with the attacher x_1 . If $d_T(x_1) = 2$, then $(A - \{x_1\}, B - \{x_0\})$ is a bipartition of the subtree $T' = T - \{x_0, x_1\}$ of T and $|A - \{x_1\}| < |B - \{x_0\}|$. It is clear that the set $A - \{x_1\}$ is a γ_{cer} -set of T' . Thus $T' \in \mathcal{T}_{\text{max}}$ and the inductive hypothesis imply that $T' \in \mathcal{T}$. Now, it remains to be proven that x_2 is not a certified critical vertex in T' . Suppose that this is not true, namely that x_2 is a certified critical vertex in T' . Then $\gamma_{\text{cer}}(T' - \{x_2\}) \leq \gamma_{\text{cer}}(T') - 1 \leq |A| - 2$. But now, if D_{cer} is a γ_{cer} -set of $T' - \{x_2\}$, then $D_{\text{cer}} \cup \{x_1\}$ is a certified dominating set of T and $|A| = \gamma_{\text{cer}}(T) \leq |D_{\text{cer}} \cup \{x_1\}| \leq |A| - 1 < \gamma_{\text{cer}}(T)$, which yields a contradiction. Consequently, since x_2 is not a certified critical vertex in T' , the tree T can be rebuilt from T' by applying the operation $\mathcal{O}_2$ with the attacher x_2 . This completes the proof. $\square$

5. k -vertex certified domination critical graphs and open problems

Let $k \in \mathbb{N}^+$. A graph G is denoted as a *k -vertex certified domination critical graph* if it is vertex certified critical and $\gamma_{\text{cer}}(G) = k$. Formally, $\gamma_{\text{cer}}(G) = k$ and for each vertex v of G , $\gamma_{\text{cer}}(G - v) \leq k - 1$. This definition indicates that 1-vertex certified domination critical graphs are the only complete graphs of order one. Now, we focus on the problem of characterising 2-vertex certified domination critical graphs. To present this characterisation, we introduce some additional terms.

A set of edges in a graph G is *independent* if no two edges in the set are adjacent in G . The edges in an independent set of edges of G are called a *matching* in G . If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M , then M is a *perfect matching* in G . Clearly, if G has a perfect matching M , then G has an even order and the edge-induced subgraph $G[M]$ is a 1-regular spanning subgraph of G . Any spanning subgraph of a graph G is referred to as a *factor* of G . A k -regular factor is referred to as a k -*factor*. For instance, a 1-factor of a graph G is the subgraph induced by a perfect matching in G . A graph G is said to be *factorable* into the factors $F_1, F_2, \dots, F_t$ if these factors are (pairwise) edge-disjoint and $\bigcup_{i=1}^t E_G(F_i) = E_G$. If G is factored into $F_1, F_2, \dots, F_t$, then $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$ is called a *factorisation* of G . If there exists a factorisation $\mathcal{F}$ of a graph G such that each factor in $\mathcal{F}$ is a k -factor (for a fixed positive integer k), then G is *k-factorable*.

It was shown in [11] that, for each positive integer n , the complete graph K_{2n} is 1-factorable. Now, let us consider a complete graph K_{2n} with a 1-factor removed. It is easy to observe that these graphs are $2n - 2$ -regular graphs, and so the following observation can be made.

Observation 5.1. *A graph G is a complete graph K_{2n} with a 1-factor removed if and only if G is $2n - 2$ -regular graph ($n \geq 1$).*

Notice that the complete graphs K_{2n} with 1-factor removed are complete graphs K_{2n} , where we remove a perfect matching. Figure 3 illustrates the complete graphs K_4 and K_6 with 1-factor removed. It is easy to check that these graphs are 2-vertex certified domination critical graphs.

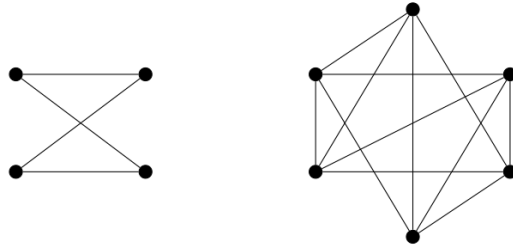


Fig. 3. Examples of complete graphs K_{2n} with 1-factor removed

Now, we characterise 2-vertex certified domination critical graphs. We start with the following observation and lemma.

Observation 5.2. *Let G be a graph. If G is a 2-vertex certified critical graph, then for each vertex v of G , the graph $G' = G - v$ contains an universal vertex.*

Lemma 5.3. *Let G be a graph of order n . If G is a 2-vertex certified domination critical graph, then G is a $n - 2$ -regular graph and n is even.*

Proof. Let G be a 2-vertex certified domination critical graph. Let D_{cer} be a γ_{cer} -set of G . Since $|D_{\text{cer}}| = 2$, let $u, v \in D_{\text{cer}}$. If u, v are shadowed vertices, then it follows that $G = K_2$ or $G = \overline{K_2}$. Now, assume that u, v are not shadowed vertices. Therefore, we may conclude that $u, v \notin S_G$ and consequently $S_G = \emptyset$. According to Theorem 3.13, we have $pn[u, D_{\text{cer}}] = \{u\}$ and $pn[v, D_{\text{cer}}] = \{v\}$. Then $uv \notin E_G$ and $N_G(u) = N_G(v)$ (otherwise, if $N_G(u) \neq N_G(v)$, then u or v must contain a private neighbour, which is a contradiction). Since every vertex of $N_G(u)$ belongs to the some γ_{cer} -set D'_{cer} and $d_G(u) = n - 2$, a similar argument as the one presented below leads to the conclusion that G is $n - 2$ -regular graph. As a consequence of the handshaking lemma, we conclude that the order n must be even. $\square$

Now, considering Lemma 5.3 and Observation 5.1, we obtain the following theorem. Notably, a similar result was mentioned in [5].

Theorem 5.4. *Let G be a graph of order n . Then G is a 2-vertex certified domination critical graph if and only if $G = K_2$, $G = \overline{K_2}$ or G is a complete graph K_{2n} with a 1-factor removed for $n \geq 1$.*

We close this paper with a list of open problems that we have yet to settle.

Problem 5.5. *Characterise the class of bipartite graphs $G = ((A, B), E_G)$, in which $\gamma_{\text{cer}}(G) = \min\{|A|, |B|\}$.*

Problem 5.6. *Characterise the class of trees T , in which $\gamma_{\text{cer}}(T) = \gamma(T)$.*

Problem 5.7. *Characterise the class of 3-vertex certified domination critical graphs.*

Problem 5.8. *Characterise the certified stable vertices and certified domination stable graphs. Recall that a vertex v of a graph G is called certified stable if $\gamma_{\text{cer}}(G - v) \geq \gamma_{\text{cer}}(G)$, while a graph G is called certified domination stable graph if all vertices of G are certified stable.*

Problem 5.9. *Identify an algorithm for checking equality $\gamma_{\text{cer}}(G) = |A|$ in bipartite graphs $G = ((A, B); E_G)$ ($1 \leq |A| \leq |B|$).*

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Research Article

On a class of graphs with equal domination and certified domination numbers

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Abstract

A graph G is called a $\mathcal{DD}_2$ -graph if its vertex set V_G can be partitioned into a dominating set D and a 2-dominating set $V_G - D$. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set. The certified domination number, $\gamma_{\text{cer}}(G)$, is the minimum size of a dominating set D such that every vertex in D has either zero or at least two neighbors in $V_G - D$. In this paper, the minimal $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$ are characterized, and it is shown that they are the cycle C_4 , the stars $K_{1,n}$ ($n \geq 2$), and the subdivisions of the connected corona graphs $\text{cor}(K_1)$ and $\text{cor}(K_2)$. A characterization is also provided for bipartite graphs in which the smaller partite set serves simultaneously as a γ -set and a γ_{cer} -set. A conjecture concerning trees and several open problems are presented in conclusion.

Keywords: dominating set; certified domination; $\mathcal{DD}_2$ -graph; minimal $\mathcal{DD}_2$ -graph; bipartite graph.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

We investigate the conditions under which the domination number $\gamma(G)$ and the certified domination number $\gamma_{\text{cer}}(G)$ of a graph G are equal, focusing on the class of $\mathcal{DD}_2$ -graphs. We generally follow the notation of [1]. For a graph $G = (V_G, E_G)$, a set $D \subseteq V_G$ is a *dominating set* if every vertex in $V_G - D$ is adjacent to a vertex in D . The minimum cardinality of such a set is the *domination number* $\gamma(G)$. A set D is *2-dominating* if every vertex in $V_G - D$ has at least two neighbors in D . A *certified dominating set* is a dominating set D where every vertex in D has zero or at least two neighbors in $V_G - D$; the minimum cardinality is the *certified domination number* $\gamma_{\text{cer}}(G)$ (see [3]). A leaf is a vertex of degree 1; its neighbor is a support vertex. The *corona graph* $\text{cor}(H)$ of a multigraph H is obtained from H by adding at least one leaf adjacent to every vertex of H . The *subdivision graph* $S(H)$ of a multigraph H is the graph obtained from H by adding a new vertex n_e to each edge $e = uv$ of H . A graph G is a $\mathcal{DD}_2$ -graph if its vertex set can be partitioned into a dominating set D and a 2-dominating set $V_G - D$ (see [4]). Such a partition is a $\mathcal{DD}_2$ -pair. As noted in [3, 6], if $(D, V_G - D)$ is a $\mathcal{DD}_2$ -pair, then D is a certified dominating set. A graph G is a *minimal $\mathcal{DD}_2$ -graph* if no proper spanning subgraph of G is a $\mathcal{DD}_2$ -graph. Miotk, Topp, and Żyliński [6] characterized connected minimal $\mathcal{DD}_2$ -graphs as being the stars $K_{1,n}$ ($n \geq 2$), the cycle C_4 , and the subdivision graphs $S(\text{cor}(H))$ of a connected corona multigraph H . In this paper, we build on this foundation to characterize the minimal $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$. We also provide a structural characterization for bipartite graphs where the smaller partite set is both a minimum dominating and minimum certified dominating set.

2. Equality condition in minimal $\mathcal{DD}_2$ -graphs

In this section, we focus on the connection between the domination number $\gamma(G)$ and the certified domination number $\gamma_{\text{cer}}(G)$, with a particular emphasis on the class of $\mathcal{DD}_2$ -graphs. We first establish a necessary condition for the equality

$$\gamma(G) = \gamma_{\text{cer}}(G)$$

and then provide a complete characterization of the connected minimal $\mathcal{DD}_2$ -graphs for which this equality holds. Let D be a dominating set of G . A vertex $v \in D$ is *shadowed* if $N_G(v) \subseteq D$. In [3], it was stated that every support vertex should be included in every γ_{cer} -set. We start by demonstrating that the equality of the two domination numbers indicates that the graph contains a particular partition that is associated with the $\mathcal{DD}_2$ property.

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Theorem 2.1. Let G be a graph without isolated vertex. The following statements are equivalent:

- (i). $\gamma(G) = \gamma_{\text{cer}}(G)$.
- (ii). There exists a γ -set D , such that the set $V_G - D$ is 2-dominating.

Proof. To prove the equivalence, we first assume that $\gamma(G) = \gamma_{\text{cer}}(G)$. Let D_{cer} be a γ_{cer} -set, which must also be a γ -set according to our assumption. For any vertex $w \in D_{\text{cer}}$, the certified property ensures that w has zero or at least two neighbors in $V_G - D_{\text{cer}}$. If any $w \in D_{\text{cer}}$ had zero neighbors in $V_G - D_{\text{cer}}$ (that is, w were shadowed by $N_G(w) \subseteq D_{\text{cer}}$), then $D' = D_{\text{cer}} - \{w\}$ would form a dominating set. Any vertex in $V_G - D_{\text{cer}}$ is dominated by $D_{\text{cer}} - \{w\}$ and since it is not adjacent to w , it must be dominated by $D_{\text{cer}} - \{w\}$, vertices in $D_{\text{cer}} - \{w\}$ would dominate themselves, and w would be dominated by its neighbors in $D_{\text{cer}} - \{w\}$. This would give $|D'| = \gamma(G) - 1$, contradicting the minimality of $\gamma(G)$. Therefore, every vertex in D_{cer} must have at least two neighbors in $V_G - D_{\text{cer}}$, which makes $V_G - D_{\text{cer}}$ a 2-dominating set, thereby proving (i) $\Rightarrow$ (ii).

Conversely, if there exists a γ -set D such that $V_G - D$ is 2-dominating, then every vertex in D has at least two neighbors outside D , making D a certified dominating set. This gives $\gamma_{\text{cer}}(G) \leq |D| = \gamma(G)$, and since $\gamma(G) \leq \gamma_{\text{cer}}(G)$ always holds by definition, we must have $\gamma(G) = \gamma_{\text{cer}}(G)$, proving (ii) $\Rightarrow$ (i). $\square$

By Theorem 2.1, if $\gamma(G) = \gamma_{\text{cer}}(G)$ (and G has no isolated vertex), then G is a $\mathcal{DD}_2$ -graph. However, the converse is not true. For example, the graph G in Figure 2.1 is a $\mathcal{DD}_2$ -graph, as the pair $(D = \{v_1, v_3, v_5, v_7, v_9\}, D_2 = \{v_0, v_2, v_4, v_6, v_8, v_{10}\})$ is a $\mathcal{DD}_2$ -pair. For this graph, a careful analysis reveals that $\gamma(G) = 4$ (with a minimum dominating set $\{v_0, v_2, v_5, v_9\}$), while $\gamma_{\text{cer}}(G) = 5$. Hence, this example demonstrates that a graph can be a $\mathcal{DD}_2$ -graph while having $\gamma(G) < \gamma_{\text{cer}}(G)$, proving that the converse of Theorem 2.1 does not hold in general.

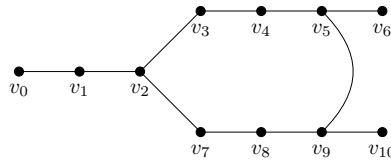


Figure 2.1: An example of a $\mathcal{DD}_2$ -graph G . Here, $\gamma(G) = 4$ and $\gamma_{\text{cer}}(G) = 5$.

This motivates the description of minimal $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$. A connected graph G is a *minimal $\mathcal{DD}_2$ -graph* if G is a $\mathcal{DD}_2$ -graph and no proper spanning subgraph of G is a $\mathcal{DD}_2$ -graph. In [6], connected minimal $\mathcal{DD}_2$ -graphs G were described as $K_{1,n}$ ($n \geq 2$), C_4 , or $G = S(\text{cor}(H))$ for some connected H . While having $\gamma(G) = \gamma_{\text{cer}}(G)$ implies that G is a $\mathcal{DD}_2$ graph (if G contains no isolated vertices), the converse is not true. We now study the structure of *minimal $\mathcal{DD}_2$ -graphs* with this equality. The following lemma considers the case $G = S(\text{cor}(H))$ and determines the structure of the γ_{cer} -set assuming equality.

Lemma 2.1. Let G be a minimal $\mathcal{DD}_2$ -graph such that $G = S(\text{cor}(H))$ for some connected graph H . If $\gamma(G) = \gamma_{\text{cer}}(G)$, then the set $D_{\text{sub}} = \{n_e : e \in E_{\text{cor}(H)}\}$ is a γ_{cer} -set of G .

Proof. Assume $G = S(\text{cor}(H))$ is a connected minimal $\mathcal{DD}_2$ -graph and that $\gamma(G) = \gamma_{\text{cer}}(G)$. Let $V_{\text{cor}(H)}$ be the vertex set of the corona graph $\text{cor}(H)$, and let $E_{\text{cor}(H)}$ be its edge set. The vertex set of G is $V_G = V_{\text{cor}(H)} \cup \{n_e : e \in E_{\text{cor}(H)}\}$, where n_e is the subdivision vertex corresponding to the edge $e \in E_{\text{cor}(H)}$. Consider the set $D_{\text{sub}} = \{n_e : e \in E_{\text{cor}(H)}\}$. We first demonstrate that D_{sub} is a certified dominating set of G . To establish *dominating property*, consider any vertex $w \in V_G \setminus D_{\text{sub}}$. This implies $w \in V_{\text{cor}(H)}$. Since H is connected and $\text{cor}(H)$ involves adding leaves to each vertex, the graph $\text{cor}(H)$ is connected and contains no isolated vertices. Thus, w must be an endpoint of at least one edge $e = wu$ in $\text{cor}(H)$. In the subdivision graph $G = S(\text{cor}(H))$, the vertex w is adjacent to the subdivision vertex n_e . As $n_e \in D_{\text{sub}}$, every vertex $w \in V_G \setminus D_{\text{sub}}$ is dominated by D_{sub} . Hence, D_{sub} is a dominating set of G . Next, we verify *certified property*. Let s be an arbitrary vertex in D_{sub} . Then $s = n_e$ for some edge $e = uv$ belonging to $E_{\text{cor}(H)}$. In the graph G , the neighbors of the subdivision vertex $s = n_e$ are precisely the endpoints of the original edge e , namely u and v . Both u and v are elements of $V_{\text{cor}(H)}$, which is equal to $V_G \setminus D_{\text{sub}}$. Therefore, the set of neighbors of s that lie outside D_{sub} is exactly $N_G(s) \cap (V_G \setminus D_{\text{sub}}) = \{u, v\}$. Since e is an edge, its endpoints u and v are distinct ($u \neq v$). Consequently, $|N_G(s) \cap (V_G \setminus D_{\text{sub}})| = 2$. Since this holds for every $s \in D_{\text{sub}}$, the set D_{sub} satisfies the condition for being a certified dominating set.

Now, consider the cardinality of D_{sub} . By definition, $|D_{sub}| = |E_{cor(H)}|$. In [6], it was established that for a connected graph H , the domination number of $G = S(cor(H))$ is precisely $\gamma(G) = |E_{cor(H)}|$. We are given the initial assumption that $\gamma(G) = \gamma_{cer}(G)$. Combining this with the known domination number, we obtain $\gamma_{cer}(G) = \gamma(G) = |E_{cor(H)}|$. Since D_{sub} is a certified dominating set and its cardinality $|D_{sub}|$ is equal to the certified domination number $\gamma_{cer}(G)$, it follows by definition that D_{sub} is a certified dominating set of minimum cardinality. Therefore, D_{sub} is a γ_{cer} -set of G . $\square$

Lemma 2.1 immediately results in the uniqueness of the minimum certified dominating set in particular graphs.

Corollary 2.1. *Let G be a minimal $\mathcal{DD}_2$ -graph such that $G = S(cor(H))$. If $\gamma(G) = \gamma_{cer}(G)$, then the set*

$$D_{sub} = \{n_e : e \in E_{cor(H)}\}$$

is the unique γ_{cer} -set of G .

Lemma 2.1 identifies the structure of the set γ_{cer} when $G = S(cor(H))$ and $\gamma(G) = \gamma_{cer}(G)$. We now establish a constraint on the base graph H that is necessary for this equality to hold. Specifically, we show that if H contains a vertex of degree two or more, the equality cannot be satisfied.

Lemma 2.2. *Let G be a minimal $\mathcal{DD}_2$ -graph $G = S(cor(H))$. If H has maximum degree $\Delta(H) \geq 2$, then $\gamma(G) < \gamma_{cer}(G)$.*

Proof. Suppose $G = S(cor(H))$ with $\Delta(H) \geq 2$, and assume that $\gamma(G) = \gamma_{cer}(G)$. By Lemma 2.1, the set

$$D_{sub} = \{n_e : e \in E_{cor(H)}\}$$

is a γ_{cer} -set, which implies $\gamma_{cer}(G) = |D_{sub}| = |E_{cor(H)}|$, and consequently, $\gamma(G) = |E_{cor(H)}|$. Now, let $v \in V_H$ with degree $d_H(v) = k \geq 2$ (such a vertex exists since $\Delta(H) \geq 2$), and let $e_{vu_1}, \dots, e_{vu_k}$ be the edges of H incident to v . Consider the set $D' = (D_{sub} - \{n_{e_{vu_1}}, \dots, n_{e_{vu_k}}\}) \cup \{v\}$, which has cardinality $|D'| = |E_{cor(H)}| - k + 1 < |E_{cor(H)}|$ since $k \geq 2$. We claim that D' is a dominating set. The vertex v dominates itself, the vertices $u_1, \dots, u_k$ and the subdivision vertices $n_{e_{vu_i}}$; any vertex $w \in V_H \setminus N_H[v]$ is dominated by some $n_{we'} \in D'$; any leaf attached to v is dominated by the corresponding subdivision vertex in D' ; any leaf attached to $u \neq v$ is similarly dominated; and any subdivision vertex not incident to v remains in D' . Therefore, D' is a dominating set with $|D'| < |E_{cor(H)}|$, which implies $\gamma(G) \leq |D'| < |E_{cor(H)}|$. This contradicts our earlier conclusion that $\gamma(G) = |E_{cor(H)}|$, proving that our initial assumption must be false, and thus $\gamma(G) < \gamma_{cer}(G)$. $\square$

Combining the known characterization of connected minimal $\mathcal{DD}_2$ -graphs with the necessary condition derived in Lemma 2.2, we arrive at the main result of this section. This theorem fully characterizes the connected minimal $\mathcal{DD}_2$ -graphs for which the domination number equals the certified domination number.

Theorem 2.2. *Let G be a connected minimal $\mathcal{DD}_2$ -graph. Then $\gamma(G) = \gamma_{cer}(G)$ if and only if G is the cycle C_4 , a star $K_{1,n}$ ($n \geq 2$) or a subdivision of $cor(K_1)$ or $cor(K_2)$.*

Proof. For a cycle C_4 , direct verification shows that any minimum dominating set of size 2 (consisting of two non-adjacent vertices) is also certified, as each vertex in the dominating set has exactly two neighbors outside the set. For a star $K_{1,n}$ with $n \geq 2$, the center vertex forms a minimum dominating set of size 1, and this vertex has $n \geq 2$ neighbors outside the set, satisfying the certified property. For $G = S(cor(K_1))$, which is a subdivision of a single vertex with at least one pendant edge, the unique subdivision vertex forms both a minimum dominating set and a minimum certified dominating set. Similarly, for $G = S(cor(K_2))$, which is a subdivision of two vertices connected by an edge and each having at least one pendant edge, the two subdivision vertices form a minimum dominating set and a minimum certified dominating set. Conversely, assume that G is a connected minimal $\mathcal{DD}_2$ -graph with $\gamma(G) = \gamma_{cer}(G)$. By the characterization theorem in [6], we know that G must be C_4 , a star $K_{1,n}$ with $n \geq 2$, or $G = S(cor(H))$ for some connected graph H . If $G = C_4$ or $G = K_{1,n}$, we are done. If $G = S(cor(H))$, then by Lemma 2.2, the equality $\gamma(G) = \gamma_{cer}(G)$ implies $\Delta(H) < 2$, which means that the maximum degree of any vertex in H is at most 1. Since H is connected, there are only two possibilities: either $H = K_1$ (a single vertex) or $H = K_2$ (two vertices connected by an edge). Therefore, G must be $S(cor(K_1))$ or $S(cor(K_2))$, which completes the proof. $\square$

Figure 2.2 presents examples of minimal $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{cer}(G)$.

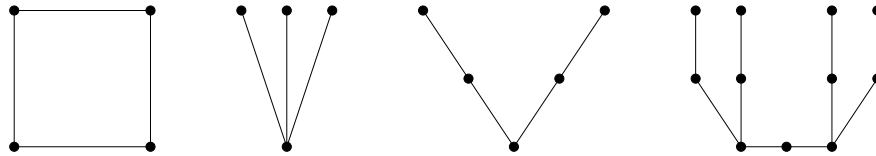


Figure 2.2: Examples of minimal $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$.

3. Certified domination relative to partite sets in bipartite graphs

This section focuses on connected bipartite graphs $G = ((A, B); E_G)$, where A and B are the partite sets with $1 \leq |A| \leq |B|$. Our primary goal is to characterize the graphs within this class for which the smaller partite set A is simultaneously a minimum dominating set (γ -set) and a minimum certified dominating set (γ_{cer} -set). This condition implies the equality

$$\gamma(G) = \gamma_{\text{cer}}(G) = |A|.$$

We build on existing characterizations of bipartite graphs where $\gamma(G) = |A|$ (see, e.g., [5, 7]) and relate this specific condition to the properties of certified domination and $\mathcal{DD}_2$ -graphs. Recall that a vertex $v \in D$ is *shadowed* with respect to D if $N_G(v) \subseteq D$. In [2] it was shown that the shadowed vertices, if they exist, must be leaves or weak support vertices. We will also utilize the structural characterization of the $\mathcal{DD}_2$ -graphs of [6].

We begin by recalling the characterization of graphs possessing the $\mathcal{DD}_2$ property, established in [6], which connects this property to certified dominating sets without shadowed vertices and the neighborhood of weak support vertices.

Lemma 3.1 (see [6]). *Let G be a graph with no isolated vertex. Then the following three properties are equivalent:*

- (i). G has a certified dominating set with no shadowed vertex.
- (ii). G is a $\mathcal{DD}_2$ -graph.
- (iii). $N_G(s) - (L_G \cup S_G) \neq \emptyset$ for every weak support s of G .

The specific structure of bipartite graphs imposes a strong constraint on shadowed vertices within a partite set. The following simple lemma clarifies this point, showing that if the set A forms a certified dominating set, it inherently satisfies one of the key conditions related to $\mathcal{DD}_2$ -graphs.

Lemma 3.2. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is the minimal certified dominating set, then A does not contain a shadowed vertex.*

Proof. A vertex $u \in A$ is shadowed with respect to A if $N_G(u) \subseteq A$. However, by the definition of a bipartite graph, if $u \in A$, then all its neighbors must be in B , i.e., $N_G(u) \subseteq B$. Therefore, $N_G(u) \cap A = \emptyset$ (unless u is isolated, which is excluded). Consequently, no vertex in A can be shadowed with respect to A in a non-trivial bipartite graph. $\square$

By combining the irrelevance of shadowing vertices in A (Lemma 3.2) with known properties of γ_{cer} -sets (that is, any γ_{cer} -set must contain all the support vertices), along with the description of $\mathcal{DD}_2$ -graphs (Lemma 3.1), we obtain straightforward necessary conditions for A being a γ_{cer} -set.

Corollary 3.1. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is a γ_{cer} -set of G , then G has the following two properties.*

- (i). $S_G \subseteq A$ or equivalently $S_G \cap B = \emptyset$;
- (ii). G is a $\mathcal{DD}_2$ -graph.

We are now ready to state the main result of this section. The following theorem gives a complete characterization of connected bipartite graphs (of order at least three) in which the smaller partite set A functions as both a γ -set and a γ_{cer} -set. The proof is structurally similar to that given for the characterization in [5] where $\gamma(G) = |A|$.

Theorem 3.1. Let $G = ((A, B); E_G)$ be a connected bipartite graph with an order of at least three such that $1 \leq |A| \leq |B|$. Then, the following statements are equivalent.

- (i). The set A is both a γ -set and a γ_{cer} -set of G .
- (ii). The graph G has the following three properties.
 - a) $S_G \cap B = \emptyset$;
 - b) G is a $\mathcal{DD}_2$ -graph;
 - c) If $x, y \in A - S_G$ and $d_G(x, y) = 2$, then there exist at least two vertices $\bar{x}, \bar{y}$, such that $N_G(\bar{x}) = N_G(\bar{y}) = \{x, y\}$.

Proof. Assume that the set A is both a γ -set and a γ_{cer} -set of G . Hence, $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$, and by Corollary 3.1 we see that G is a $\mathcal{DD}_2$ -graph and $S_G \cap B = \emptyset$. This proves properties a) and b). Now, let $x, y \in A - S_G$ for which $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$. If there exists exactly one vertex z , then the set $A - \{x, y\} \cup \{z\}$ would be a smaller dominating set, which is a contradiction with equality $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$. Therefore, it follows that there exists a vertex $\bar{x} \in B - \{z\}$ such that $N_G(\bar{x}) \subseteq \{x, y\}$. Since $x, y \notin S_G$ and $\bar{x}$ is not a support vertex, it follows that $N_G(\bar{x}) = \{x, y\}$ (otherwise if $d_G(\bar{x}) > 2$, then the set $A - N_G(\bar{x}) \cup \{\bar{x}\}$ would be a smaller dominating set, which contradicts the equality $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$). Hence, there exists $\bar{y} \in B$ for which $N_G(\bar{y}) = \{x, y\}$. This proves property c).

Now, assume that the graph G has the three properties a), b), and c). First, it suffices to show that the set A is a γ -set of G . We claim that $\gamma(G) = |A|$. Since the set A is a dominating set, we see that $\gamma(G) \leq |A|$. Now, it suffices to show that $\gamma(G) \geq |A|$. Suppose, for contradiction, that $\gamma(G) < |A|$. Let D be a γ -set of G , such that $|D \cap A|$ is as large as possible (if $D \subset A$, then the set D would not be dominating in G). Since $|A - D| > |B \cap D| \geq 1$ and from the fact that every vertex in $A - D$ has at least one neighbor in $B \cap D$ we see that there exists two vertices $x, y \in A$ with common neighbor $z \in B$ (otherwise, we obtain a larger $A \cap D$, a contradiction). Since $S_G \subseteq A \cap D$ and $x, y \notin L_G$ we have $x, y \in A - S_G$. Then, by property c) it follows that there exists vertices $\bar{x}, \bar{y} \in B$, such that $N_G(\bar{x}) = N_G(\bar{y}) = \{x, y\}$. Then the set $D' = D - \{z\} \cup \{x, y\}$ is the dominating set, a contradiction with maximality of $|D \cap A|$. Hence, $\gamma(G) = |A|$ and the set A is a γ -set of G . Now we show that the set A is also a certified dominating set. Since G is a connected graph on at least three vertices and from a), it follows that every support vertex has at least two neighbors in the set B . From c) we have it that for every two non-support vertices x, y belong to A there exists at least two neighbors in the set B . Since $\gamma(G) = |A|$ and for every vertex $v \in A$, we have it that $|N_G(v) \cap (V_G - A)| \geq 2$. This implies that the set A is also a certified dominating set. Consequently, A is both a γ -set and a γ_{cer} -set of G . $\square$

Figure 3.1 shows an example of a bipartite graph G with partition $A = \{v_1, v_3, v_5, v_7, v_9\}$ and $B = \{v_0, v_2, v_4, v_6, v_8, v_{10}\}$. The set $D = \{v_1, v_3, v_5, v_7, v_9\}$ forms a certified dominating set because each vertex in D has exactly two neighbors outside D , which satisfies the certified property. Careful examination shows that the set $\{v_1, v_2, v_5, v_9\}$ forms a γ -set of G , and hence, $\gamma(G) = 4$. However, no certified dominating set smaller than D exists because any attempt to use fewer vertices would result in at least one vertex having exactly one neighbor outside the set, violating the certified condition. Thus, $\gamma_{\text{cer}}(G) = |D| = 5 = |A|$, demonstrating that

$$\gamma(G) < \gamma_{\text{cer}}(G) = |A|.$$

This example demonstrates that a graph can be a $\mathcal{DD}_2$ -graph while having $\gamma(G) < \gamma_{\text{cer}}(G)$; thus, the converse of Theorem 2.1 does not hold in general.

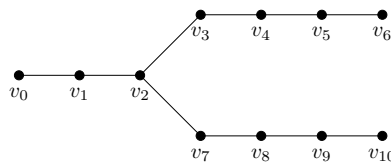


Figure 3.1: A bipartite $\mathcal{DD}_2$ -graph $G = ((A, B); E_G)$ where the smaller partite set $A = \{v_1, v_3, v_5, v_7, v_9\}$ is a γ_{cer} -set, but not γ -set, demonstrating that $\gamma(G) < \gamma_{\text{cer}}(G) = |A|$.

4. Concluding remarks and open problems

In [6], it was observed that a tree T is a $\mathcal{DD}_2$ -graph if and only if T can be decomposed into a forest F where each component is a star $K_{1,n}$ ($n \geq 2$) or the subdivision graph of a corona tree. By Theorem 2.2, it follows that if this decomposition contains a component of the form $S(\text{cor}(H))$, where H is not K_1 or K_2 , then $\gamma(T) < \gamma_{\text{cer}}(T)$. Hence, we have the following conjecture.

Conjecture 4.1. *Let T be a tree. Then $\gamma(T) = \gamma_{\text{cer}}(T)$ if and only if in the structural decomposition of T related to $\mathcal{DD}_2$ -graphs described in [6], every component is a star $K_{1,n}$ ($n \geq 2$) or is a subdivision of $\text{cor}(K_1)$ or $\text{cor}(K_2)$.*

We end this paper with the following open problems.

Problem 4.1. *Determine the class of $\mathcal{DD}_2$ -graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$.*

Problem 4.2. *Determine the class of bipartite $\mathcal{DD}_2$ -graphs $G = ((A, B); E_G)$ for which $\gamma_{\text{cer}}(G) = |A|$, where $1 \leq |A| \leq |B|$.*

Problem 4.3. *Determine an algorithm to check the equality $\gamma_{\text{cer}}(G) = |A|$ in bipartite graphs $G = ((A, B); E_G)$, where $1 \leq |A| \leq |B|$.*

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Disjoint dominating and 2-dominating sets in graphs

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ABSTRACT

A subset $D \subseteq V_G$ is a dominating set of G if every vertex in $V_G - D$ has a neighbor in D , while D is 2-dominating if every vertex in $V_G - D$ has at least two neighbors in D . A graph G is a DD_2 -graph if it has a pair (D, D_2) of disjoint subsets of vertices such that D is a dominating set, and D_2 is a 2-dominating set of G . Studies of first properties of the DD_2 -graphs were initiated by Henning and Rall (2013). In this paper, we continue their study and complete their structural characterization of all DD_2 -graphs. Next, we focus on minimal DD_2 -graphs and provide the relevant characterization of that class of graphs as well. Additionally, we study optimization problems related to DD_2 -graphs and non- DD_2 -graphs, respectively. In particular, for a given DD_2 -graph G , the purpose is to find a minimal spanning DD_2 -graph of G of minimum or maximum size. We show that both these problems are NP-hard. Finally, if G is a graph which is not a DD_2 -graph, we consider the question of how many edges must be added to G or subdivided in G to ensure the existence of a DD_2 -pair in the resulting graph. The latter problem turned out to be polynomially tractable, while the former one is NP-hard.

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1. Introduction

Let $G = (V_G, E_G)$ be a graph. A set of vertices $D \subseteq V_G$ of G is *dominating* if every vertex in $V_G - D$ has a neighbor in D , while D is *2-dominating* if every vertex in $V_G - D$ has at least two neighbors in D . A set $D \subseteq V_G$ is a *total dominating set* if every vertex has a neighbor in D . A set $D \subseteq V_G$ is a *paired-dominating set* if D is a total dominating set and the subgraph induced by D contains a perfect matching. Ore [1] was the first to observe that a graph with no isolated vertex contains two disjoint dominating sets. Consequently, the vertex set of a graph without isolated vertices can be partitioned into two dominating sets. Various graph theoretic properties and parameters of graphs having disjoint dominating sets are studied in [2–8]. Characterizations of graphs with disjoint dominating and total dominating sets are given in [9–15], while in [16–20] graphs which have the property that their vertex set can be partitioned into two

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disjoint total dominating sets are studied. Conditions which guarantee the existence of a dominating set whose complement contains a 2-dominating set, a paired-dominating set or an independent dominating set are presented in [7,10,14,21–23]. In this paper we first restrict our attention to conditions which ensure a partition of vertex set of a graph into a dominating set and a 2-dominating set. The study of graphs having a dominating set whose complement is a 2-dominating set has been initiated by Henning and Rall [22]. They define a DD_2 -pair in a graph G to be a pair (X, Y) of disjoint sets of vertices of G such that X is a dominating set, and Y is a 2-dominating set of G . A graph that has a DD_2 -pair is called a DD_2 -graph. It is easy to observe that a complete graph K_n is a DD_2 -graph if $n \geq 3$, a path P_n is a DD_2 -graph if and only if $n = 3$ or $n \geq 5$, while a cycle C_n is a DD_2 -graph for every $n \geq 3$. On the other hand, if G is a graph obtained by adding a pendant edge to each vertex of an arbitrary graph F , then G is not a DD_2 -graph, but if H is a graph obtained by adding at least two pendant edges to each vertex of an arbitrary graph F , then H is a DD_2 -graph. Henning and Rall [22] observed that every graph with minimum degree at least two is a DD_2 -graph. They also provided a constructive characterization of trees that are DD_2 -graphs, and established that the complete bipartite graph $K_{3,3}$ is the only connected graph with minimum degree at least three for which $D \cup D_2$ necessarily contains all vertices of the graph. Herein, we continue their study and complete their structural characterization of all DD_2 -graphs. Next, we focus on minimal DD_2 -graphs (which belong to the class of edge-critical graphs, intensively studied in graph theory, see for example Chapter 5 of [24] and Chapters 16 and 17 of [25]) and provide the relevant characterization of that class of graphs as well. All these results have also algorithmic consequences, leading to simple linear time algorithms for recognizing the two aforementioned graph classes. Next, we study optimization problems related to DD_2 -graphs and non- DD_2 -graphs, respectively. In particular, for a given DD_2 -graph G , the purpose is to find a minimal spanning DD_2 -graph of G of minimum or maximum size. We show that both these problems are NP-hard. Finally, if G is a graph which is not a DD_2 -graph, we consider the question of how many edges must be added to G or subdivided in G to ensure the existence of a DD_2 -pair in the resulting graph. The latter problem turned out to be polynomially tractable, while the former one is NP-hard.

For notation and graph theory terminology we in general follow [26]. Specifically, for a vertex v of a graph $G = (V_G, E_G)$, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v , and the cardinality of $N_G(v)$, denoted by $d_G(v)$, is called the *degree* of v . The *closed neighborhood* of v , denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, for a subset $X \subseteq V_G$ of vertices, the *neighborhood* of X , denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the *closed neighborhood* of X , denoted by $N_G[X]$, is the set $N_G(X) \cup X$. The minimum degree of a vertex in G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf*, and the only neighbor of a leaf is called its *support vertex* (or simply, its *support*). If a support vertex has at least two leaves as neighbors, we call it a *strong support*, otherwise it is a *weak support*. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of G is denoted by L_G , S'_G , S''_G , and S_G , respectively.

2. Structural characterization of DD_2 -graphs

In this section we present a structural characterization of DD_2 -graphs. We begin with three useful preliminary results.

Observation 2.1 ([22]). *If (D, D_2) is a DD_2 -pair in a graph G , then every leaf of G belongs to D_2 , while every support of G belongs to D , that is, $L_G \subseteq D_2$ and $S_G \subseteq D$.*

Observation 2.2. *A graph G is a DD_2 -graph if and only if G has a spanning bipartite subgraph $H = (A, B, E_H)$ such that $d_H(a) \geq 2$ for every $a \in A$, while $d_H(b) \geq 1$ for every $b \in B$.*

Proof. Let (D, D_2) be a DD_2 -pair in G . Then the bipartite graph $H = (A, B, E_H)$, where $A = D$, $B = V_G - D$, and $E_H = E_G(D, V_G - D)$ is the set of edges that join a vertex of D and a vertex of $V_G - D$, is the desired spanning subgraph of G .

On the other hand if $H = (A, B, E_H)$ is a bipartite spanning subgraph of G such that $d_H(a) \geq 2$ for every $a \in A$, and $d_H(b) \geq 1$ for every $b \in B$, then (A, B) is a DD_2 -pair in H , and, therefore, in G . $\square$

From [Observation 2.2](#) (or directly from the definition of a DD_2 -graph) we immediately have the following corollary.

Corollary 2.3. *Every spanning supergraph of a DD_2 -graph is a DD_2 -graph.*

Before we state and prove our key characterization of DD_2 -graphs we need some more terminology concerning dominating sets. A dominating set D of a graph G is said to be *certified* if every vertex in D has either zero or at least two neighbors in $V_G - D$ (see [\[27,28\]](#)). A vertex v of a graph G is said to be *shadowed* with respect to a certified dominating set D if $N_G[v] \subseteq D$. In the next theorem we prove that the DD_2 -graphs are precisely the graphs having a certified dominating set with no shadowed vertex. We also prove that a graph G is a DD_2 -graph if and only if the neighborhood of each weak support of G contains a vertex which is neither a leaf nor a support vertex.

Theorem 2.4. *Let G be a graph with no isolated vertex. Then the following three properties are equivalent:*

- (1) G has a certified dominating set with no shadowed vertex.
- (2) G is a DD_2 -graph.
- (3) $N_G(s) - (L_G \cup S_G) \neq \emptyset$ for every weak support s of G .

Proof. The implication (1) $\Rightarrow$ (2) was already observed in [\[27\]](#). However, for clarity, we repeat the arguments. Let D be a certified dominating set of G and assume that no element of D is shadowed. Then D is a dominating set of G and $|N_G(x) \cap (V_G - D)| \geq 2$ for every $x \in D$. Consequently, the sets D and $V_G - D$ form a DD_2 -pair in G , and so G is a DD_2 -graph.

Assume now that (D, D_2) is a DD_2 -pair of G , and let s be a weak support of G . Then $L_G \subseteq D_2$ and $S_G \subseteq D$ (by [Observation 2.1](#)), and $s \in S'_G \subseteq S_G \subseteq D \subseteq V_G - D_2$. Consequently $N_G(s) \cap (D_2 - L_G) \neq \emptyset$ (since $|N_G(s) \cap L_G| = 1$ and $|N_G(s) \cap D_2| \geq 2$). This implies that the set $N_G(s) - (L_G \cup S_G)$ is nonempty (as its subset $N_G(s) \cap (D_2 - L_G)$ is nonempty) and this establishes the implication (2) $\Rightarrow$ (3).

Finally, assume that $N_G(s) - (L_G \cup S_G) \neq \emptyset$ for every weak support s of G . Let I be a maximal independent subset of $V_G - N_G[S_G]$ in G . We claim that $I \cup S_G$ is the desired certified dominating set with no shadowed vertex in G . The choice of I and the definition of $N_G[S_G]$ imply that $I \cup S_G$ is a dominating set in G . Thus it remains to show that $|N_G(x) - (I \cup S_G)| \geq 2$ for $x \in I \cup S_G$. This is obvious if $x \in I$ or if x is a strong support. If x is a weak support, then also $|N_G(x) - (I \cup S_G)| \geq 2$, as in this case x is adjacent to exactly one leaf (and $L_G \subseteq V_G - (I \cup S_G)$) and, by the assumption, it has another neighbor in $N_G[S_G] - (L_G \cup S_G) \subseteq V_G - (I \cup S_G)$. This completes the proof of the implication (3) $\Rightarrow$ (1). $\square$

A graph with minimum degree at least two has no (weak) support vertex and therefore the next corollary is obvious from [Theorem 2.4](#).

Corollary 2.5 ([\[22\]](#)). *Every graph with minimum degree at least two is a DD_2 -graph.*

Remark 1. By [Theorem 2.4](#), a necessary and sufficient condition for a graph to be a DD_2 -graph is that each weak support has a neighbor which is neither a leaf nor a support vertex. Thus, since identifying all

leaves and support vertices in a graph of order n and size m can be done in $O(n + m)$ time, the problem of recognizing whether a given graph is a DD_2 -graph can be solved in linear time.

A connected graph G is said to be a *minimal DD_2 -graph*, if G is a DD_2 -graph and no proper spanning subgraph of G is a DD_2 -graph. We say that a disconnected graph G is a *minimal DD_2 -graph* if every connected component of G is a minimal DD_2 -graph. A multigraph H is called a *corona graph* if every vertex of H is a leaf or it is adjacent to a leaf of H . The *subdivision graph* $S(H)$ of a multigraph H is the graph obtained from H by inserting a new vertex onto each edge of H . If e is an edge of H , then by n_e we denote the vertex inserted onto e in $S(H)$. We describe the structure of minimal DD_2 -graphs in the following theorem.

Theorem 2.6. *A connected graph G is a minimal DD_2 -graph if and only if G is a star $K_{1,n}$ ($n \geq 2$), a cycle C_4 , or G is the subdivision graph of a connected corona graph, that is, $G = S(H)$ for some connected corona multigraph H .*

Proof. It is easy to check that $K_{1,n}$ ($n \geq 2$) and C_4 are minimal DD_2 -graphs. Now let H be a connected corona multigraph. Then from the equivalence of the statements (2) and (3) of Theorem 2.4 it follows immediately that $S(H)$ is a DD_2 -graph. Let F be a DD_2 -graph which is a spanning subgraph of $S(H)$. To prove the minimality of $S(H)$ it suffices to show that every edge of $S(H)$ is in F . First, if vu is a pendant edge in H , then, since no component of F is of order 1 or 2, each of the edges vn_{vu} and un_{vu} is in F . Thus assume that vu is an edge in H and no one of the vertices v and u is a leaf in H . Then v and u are supports in H , and, therefore, there exist vertices v' and u' such that vv' and uu' are pendant edges in H . Since n_{vu} is not an isolated vertex in F , at least one of the edges vn_{vu} and un_{vu} is in F . It remains to prove that the case where exactly one of the edges vn_{vu} and un_{vu} is in F is impossible. Without loss generality assume that only un_{vu} is in F . Then n_{vu} is a leaf in F , u is a support in F , but now $N_F(n_{uu'}) \subseteq L_F \cup S_F$, and it follows from Theorem 2.4 that F is not a DD_2 -graph. This contradiction proves the minimality of $S(H)$.

Assume now that G is a connected minimal DD_2 -graph. Then, by Observation 2.2, G is a bipartite graph, say $G = (A, B, E_G)$, and without loss of generality we may assume that $d_G(a) \geq 2$ for every $a \in A$, and $d_G(b) \geq 1$ for every $b \in B$. We now consider the following two cases.

Case 1: $d_G(a) \geq 3$ for some $a \in A$. If $d_G(b) = 1$ for every $b \in N_G(a)$, then G is a star, $G = K_{1,n}$ ($n = d_G(a)$). It remains to observe that the case $d_G(a) \geq 3$ (for some $a \in A$) and $d_G(b) \geq 2$ for some $b \in N_G(a)$ is impossible, as otherwise it immediately follows from Observation 2.2 that the proper spanning subgraph $G' = G - ab$ of G would be a DD_2 -graph, contradicting the minimality of G .

Case 2: $d_G(a) = 2$ for every $a \in A$. Then the set A can be divided into the sets $A_{xy} = N_G(x) \cap N_G(y)$, where $x, y \in B$ and the distance $d_G(x, y) = 2$. Now it is straightforward to observe that G is the subdivision graph of the multigraph $H = (V_H, E_H)$ in which $V_H = B$ and there is a one-to-one correspondence between the edges joining vertices x and y in H and the elements of the set A_{xy} in G . If $|A| = 1$, then $G = K_{1,2}$ and it is the subdivision graph of a corona graph, as $K_{1,2} = S(P_2) = S(K_1 \circ K_1)$. If $|A| = 2$, then it follows from the connectivity of G that $G = C_4$ or $G = P_5$ and the last graph is the subdivision graph of a corona graph, as $P_5 = S(P_3) = S(K_1 \circ 2K_1)$. Thus assume that $|A| \geq 3$. Now it remains to prove that H is a corona graph. Suppose to the contrary that $V_H - (L_H \cup S_H) \neq \emptyset$. We consider two subcases.

Subcase 2.1. Assume first that there exists $x \in V_H - (L_H \cup S_H)$ such that $N_H(y) - \{x\} \neq \emptyset$ for every $y \in N_H(x)$, in other words, $d_G(y) - |A_{xy}| \geq 1$ for every $y \in N_H(x)$. In this case $d_H(x) > 1$ (as x is not a leaf) and $G' = G - \bigcup_{y \in N_H(x)} \{yt : t \in A_{xy}\}$ is a proper spanning subgraph of G . This graph consists of two vertex-disjoint graphs $G_1 = G[N_G[x]]$ and $G_2 = G - N_H[x]$. The fact that G_1 is a star of order at least 3 implies that G_1 is a DD_2 -graph, while the fact that G_2 is a DD_2 -graph follows from Observation 2.2 and from the choice of x (implying that $d_{G_2}(a) = d_G(a) = 2$ for $a \in A \cap V_{G_2}$, and $d_{G_2}(b) = d_G(b) - |A_{xb}| \geq 1$ for $b \in V_{G_2} \cap B$). Consequently, $G' = G_1 \cup G_2$ is a DD_2 -graph, which contradicts the minimality of G .

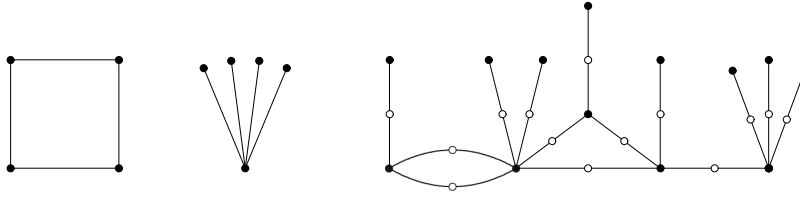


Fig. 1. A disconnected minimal DD_2 -graph.

Subcase 2.2. Assume now that for every $x \in V_H - (L_H \cup S_H)$ there exists $y \in N_H(x)$ such that $N_H(y) = \{x\}$. Take any $x_0 \in V_H - (L_H \cup S_H)$ and $y_0 \in N_H(x_0)$ such that $N_H(y_0) = \{x_0\}$. In this case there is a multiple edge between x_0 and y_0 . If $N_H(x_0) = \{y_0\}$, then $G = K_{2,n}$ ($n = |A| \geq 3$ is the number of edges joining x_0 and y_0) and it is a non-minimal DD_2 -graph. If $N_H(x_0) \neq \{y_0\}$, then the same arguments as in Subcase 2.1 prove that the proper spanning subgraph $G' = G - \{x_0 t : t \in A_{x_0 y_0}\}$ of G is a non-minimal DD_2 -graph. This completes the proof. $\square$

Fig. 1 shows a disconnected minimal DD_2 -graph that consists of three types of connected minimal DD_2 -graphs. The last one is the subdivision graph of a corona graph, and one can observe that the main properties of such graphs may be rephrased as in the next observation, which we present without any proof.

Observation 2.7. *A connected graph G is the subdivision graph of a corona graph if and only if G is a bipartite graph, say $G = (A, B, E_G)$, such that $d_G(a) = 2$ for every $a \in A$, while every $b \in B$ is a leaf or it is at the distance two from some leaf of G .*

Remark 2. It follows from Observation 2.2, Corollary 2.3, and Theorem 2.6 that a graph G is a DD_2 -graph if and only if G is a spanning supergraph of a minimal DD_2 -graph, that is, G is a spanning supergraph of a graph in which every connected component is a cycle C_4 , a star $K_{1,n}$ (with $n \geq 2$), or the subdivision graph of a corona graph. In fact, it is possible to show a little more: If G is a DD_2 -graph and C_4 is not a connected component of G , then G is a spanning supergraph of a graph in which every connected component is a star $K_{1,n}$ ($n \geq 2$) or the subdivision graph of a corona graph. Therefore, a tree T is a DD_2 -graph if and only if T has a spanning forest F in which every connected component is a star $K_{1,n}$ ($n \geq 2$) or the subdivision graph of a corona tree. Such trees were also constructively characterized in [22], using vertex labeling. (Taking into account Observation 2.7, we point out that if for every connected component of F , each of its leaves as well as each vertex at the distance two to a leaf is assigned the label B , while any other vertex is assigned the label A , then the resulting vertex labeling of F and so of T as well is the one discussed in [22].)

Remark 3. It follows from Theorem 2.6 that recognizing connected minimal DD_2 -graphs is relatively easy. Let G be a connected graph of order n and size m . First we check whether $G = C_4$ or $G = K_{1,n}$ for some $n \geq 2$. If G is neither C_4 nor $K_{1,n}$, then we check the bipartiteness of G and ad hoc determine the degrees of the vertices in G . If G is a non-bipartite graph or $\delta(G) \geq 2$, then G is not a minimal DD_2 -graph (as it follows from Theorem 2.6). Thus assume that G is a bipartite graph and $\delta(G) = 1$. Let (A, B) be the bipartition of G . If both $A \cap L_G$ and $B \cap L_G$ are nonempty sets, then G is not a DD_2 -graph (by Observation 2.2). Therefore without loss of generality we may assume that $L_G \subseteq B$. Now, since $G \neq K_{1,n}$, it must be $d_G(a) = 2$ for every $a \in A$ (as it was observed in Case 1 of the proof of Theorem 2.6). By Observation 2.7, it remains to check whether every vertex in $B - L_G$ is at the distance two from some leaf of G . Since each of the above steps can be done in $O(n + m)$ time, we conclude that the problem of recognizing whether a given graph G is a minimal DD_2 -graph can be solved in linear time.

Remark 4. If G is a DD_2 -graph, then let $\gamma\gamma_2(G)$ denote the integer $\min\{|D|+|D_2| : (D, D_2) \text{ is a } DD_2\text{-pair in } G\}$. This parameter has been defined and studied in [22]. It is obvious that if G is a DD_2 -graph, then $3 \leq \gamma\gamma_2(G) \leq |V_G|$. It is also easy to observe that if n and k are integers such that $3 \leq k \leq n$ and $G_{n,k}$ is one of the graphs $K_1 + ((k-3)K_1 \cup K_{n+2-k})$ and $K_1 + ((k-3)K_1 \cup K_{2,n-k})$, then $|V_{G_{n,k}}| = n$ and $\gamma\gamma_2(G_{n,k}) = k$. In addition, if G is a graph of order n , then $\gamma\gamma_2(G) = 3$ if and only if $K_1 + K_{2,n-3}$ is a spanning subgraph of G . Henning and Rall in their paper [22] studied graphs G for which $\gamma\gamma_2(G) = |V_G|$. In particular, they observed that there are infinitely many graphs G with minimum degree two for which $\gamma\gamma_2(G) = |V_G|$, and proved that $G = K_{3,3}$ is the only graph with $\delta(G) \geq 3$ for which $\gamma\gamma_2(G) = |V_G|$. A complete characterization of such graphs remains an open problem. Nevertheless, herein, by establishing the following theorem, we make a small step forward.

Theorem 2.8. *If G is a minimal DD_2 -graph, then $\gamma\gamma_2(G) = |V_G|$.*

Proof. Without loss generality assume that G is connected. Then, by Theorem 2.6, $G = K_{1,n}$ ($n \geq 2$), $G = C_4$, or $G = S(H)$ for some connected corona graph H . The statement is obvious if $G = K_{1,n}$ ($n \geq 2$) or $G = C_4$. Thus assume that $G = S(H)$, where H is a connected corona graph. Let (D, D_2) be a DD_2 -pair in G . If vu is a pendant edge in H , then we must have $\{v, u\} \subseteq D_2$ and $n_{vu} \in D$. This implies that $V_H \subseteq D$. Finally, if ab is an inner edge in H , then $n_{ab} \in D$ (as D is a dominating set in $G = S(H)$ and no neighbor of n_{ab} is in D). This proves that $D_2 = V_H$ and $D = \{n_{xy} : xy \in E_H\}$. Therefore $\gamma\gamma_2(G) = |D|+|D_2| = |V_G|$. $\square$

3. Spanning minimal DD_2 -graphs of a graph

It is obvious that a graph may have many non-isomorphic spanning minimal DD_2 -graphs. For example, the complete graph $G = K_{3n}$ ($n \geq 3$) has a spanning minimal DD_2 -graph being $S(H \circ K_1)$ of size $6n - 8$, where H is the multigraph of order 2 in which the only two vertices are joined by $3n - 6$ edges (so being maximal in the number of edges over all its spanning minimal DD_2 -graphs), and it also has a spanning minimal DD_2 -graph that consists of n disjoint 3-vertex paths of size $2n$ (so being minimal in the number of edges over all its spanning minimal DD_2 -graphs). Therefore, for a given graph G , a natural computational problem is to determine a spanning minimal DD_2 -graph of the minimum size. Observe that any minimal DD_2 -graph has at least three vertices and the 3-vertex path is a minimal DD_2 -graph. Therefore, a spanning minimal DD_2 -graph of a graph G of order n must be of size at least $2n/3$. Since the relevant perfect P_3 -matching problem – where for a given graph G , the problem is to determine whether all the vertices of G can be simultaneously covered by disjoint 3-vertex paths in G – is NP-complete even for cubic bipartite planar 2-connected graphs [29], we immediately conclude with the following theorem.

Theorem 3.1. *The problem of determining a spanning minimal DD_2 -graph of the minimum size is NP-hard even for cubic bipartite planar 2-connected graphs.*

As regards the maximization variant, the problem also remains NP-hard. The idea of our proof is standard and it is based upon reduction from the restricted variant of the 3-dimensional matching problem [30,31], being a NP-hard problem [32].

Problem 3.2 (The 3DM3 Problem). Let $G = (V \cup U, E)$ be a subcubic bipartite planar graph with no leaf, where V is the union of disjoint sets X, Y , and Z , where $|X| = |Y| = |Z| = q$, and every vertex $u \in U$ is adjacent to exactly one vertex from each of the sets X, Y , and Z . Is there a subset $U' \subseteq U$ of cardinality q dominating all vertices in V ?

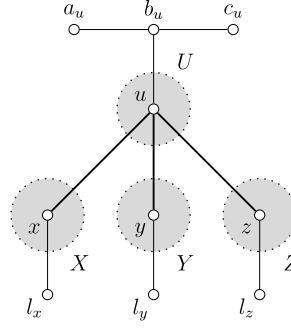


Fig. 2. Idea of the construction of the supergraph G^s of G .

Next, let us define (the decision version of) the *maximum spanning minimal DD_2 -graph problem* (the *Max- DD_2* problem for short).

Problem 3.3 (*The Max- DD_2 Problem*). Let G be a bipartite planar DD_2 -graph, and let k be a positive integer. Does G have a spanning minimal DD_2 -graph of size k ?

Let $G = (V \cup U, E)$ be a subcubic bipartite planar graph with no leaf, where V is the union of disjoint sets X, Y , and Z , where $|X| = |Y| = |Z| = q$, and every vertex $u \in U$ is adjacent to exactly one vertex from each of the sets X, Y , and Z . Let $G^s = (V_{G^s}, E_{G^s})$ be the supergraph of G obtained from G by adjoining to each vertex $v \in X \cup Y \cup Z$, separately, the leaf l_v , and to each vertex $u \in U$ — the three-vertex path $a_u b_u c_u$, as illustrated in Fig. 2. Formally, G^s is the graph in which $V_{G^s} = V \cup \{l_v : v \in V\} \cup \bigcup_{u \in U} \{a_u, b_u, c_u\}$, and $E_{G^s} = E \cup E_V \cup E_U$, where $E_V = \{vl_v : v \in V\}$ and $E_U = \bigcup_{u \in U} \{a_u b_u, b_u c_u, a_u c_u\}$. Clearly, the supergraph G^s has $2|V| + 4|U|$ vertices and $|V| + 2|E| = |V| + 6|U|$ edges, it remains planar, bipartite, and $\Delta(G^s) = 4$. Furthermore, by Theorem 2.4, G^s is a DD_2 -graph (as each weak support of G^s has a neighbor which is neither a leaf nor a support).

Lemma 3.4. *There exists a solution to the 3DM3 problem in G if and only if there exists a solution to the Max- DD_2 problem in the supergraph G^s with the parameter $k = |V| + 3|U| + 2q$.*

Proof. Let U' with $|U'| = q$ be a solution to the 3DM3 problem in $G = (V \cup U, E)$. A spanning minimal DD_2 -graph of $G^s = (V^s, E^s)$ consists of the following $|U| + q$ components F_u^3, F_u^7, T_w^4 , all being trees (see Fig. 2), where for $u \in U'$ and for $w \in U - U'$:

- $F_u^3 = (V_u^3, E_u^3)$, where $V_u^3 = \{a_u, b_u, c_u\}$, $E_u^3 = \{a_u b_u, b_u c_u\}$;
- $F_u^7 = (V_u^7, E_u^7)$, $V_u^7 = N_G[u] \cup \{l_v : v \in N_G(u)\}$, $E_u^7 = \bigcup_{v \in N_G(u)} \{uv, vl_v\}$;
- $T_w^4 = (V_w^4, E_w^4)$, where $V_w^4 = \{w, a_w, b_w, c_w\}$, $E_w^4 = \{wb_w, a_w b_w, b_w c_w\}$.

Clearly, each of the graphs F_u^3 , F_u^7 , and T_w^4 is a minimal DD_2 -graph, and all these graphs constitute a spanning subgraph of G^s of size $8q + 3|U - U'| = 5q + 3|U| = |V| + 3|U| + 2q$ as required.

On the other hand, let $H = (V_H, E_H)$ be a spanning minimal DD_2 -graph of size $|V| + 3|U| + 2q$ of G^s . The following claims are consequences of the structure of G^s and Theorem 2.6.

Claim 1. *For a vertex $v \in V$, since v is a weak support in G^s , we have $2 \leq d_H(v) \leq 4$ and $vl_v \in E_H$. Furthermore, if $d_H(v) \geq 3$, then the connected component of H which v belongs to is isomorphic to either $K_{1,3}$ or $K_{1,4}$.*

Claim 2. H is acyclic and has exactly $|U| + q$ connected components.

Proof. Suppose that a connected component C of H has a cycle. Then $d_C(v) \geq 3$ for some $v \in V$, immediately implying that $C = K_{1,3}$ or $C = K_{1,4}$ by Claim 1, a contradiction. Next, since H is a forest of order $2|V| + 4|U|$ and size $|V| + 3|U| + 2q$, H has $(2|V| + 4|U|) - (|V| + 3|U| + 2q) = |U| + q$ connected components (as $|V| = 3q$). $\square$

Claim 3. All leaves in $\{l_v : v \in V\}$ belong to at most q components of H .

Proof. For a vertex $u \in U$, since b_u is a strong support in G^s , we have $2 \leq d_H(b_u) \leq 3$, both edges $a_u b_u$ and $b_u c_u$ belong to E_H , and the connected component of H which b_u belongs to is isomorphic to either P_3 or $K_{1,3}$ (by Theorem 2.6). Therefore, H has at least $|U|$ connected components such that no vertex in V belongs to any of them. Consequently, taking into account Claim 2, all leaves in $\{l_v : v \in V\}$ belong to at most q components of H (we shall refer to those components as l -components). $\square$

Claim 4. For any two vertices $x', x'' \in X$, the leaves $l_{x'}$ and $l_{x''}$, and so both x' and x'' , belong to two distinct l -components of H . The analogous properties hold for any two vertices $y', y'' \in Y$ and any two vertices $z', z'' \in Z$.

Proof. Suppose that C is an l -component of H which both $l_{x'}$ and $l_{x''}$ belong to. Then the diameter of C is at least six, and $d_C(v) \geq 3$ for some $v \in V$, immediately implying that $C = K_{1,3}$ or $C = K_{1,4}$ by Claim 1, a contradiction. $\square$

Now, taking into account the structure of the graph G , in particular, the fact that $d_G(u) = 3$, by combining Claims 1, 3 and 4, we may conclude that there are exactly q l -components in H , say $C_1, \dots, C_q$, such that each component C_i has exactly one vertex from each of the sets X, Y, Z and U , say x_i, y_i, z_i and u_i , $i = 1, \dots, q$. Since $N_G(u_i) = \{x_i, y_i, z_i\}$ for $i = 1, \dots, q$, the set $U' = \{u_1, \dots, u_q\}$ constitutes a solution to the 3DM3 problem in G . $\square$

Clearly, the above reduction takes polynomial time (in the order and the size of a graph G). Also, a non-deterministic polynomial algorithm for the Max- DD_2 -problem in G just guesses an edge-cover C of G and checks whether C is a DD_2 -graph (which can be done in polynomial time by Remark 1). Hence by Lemma 3.4 and the fact that the 3DM3 problem is NP-complete [32], we conclude with the following theorem.

Theorem 3.5. The Max- DD_2 problem in bipartite planar graphs of maximum degree at most four is NP-complete. Consequently, the problem of determining the spanning minimal DD_2 -graph of maximum size of bipartite planar graphs of maximum degree at most four is NP-hard.

4. DD_2 -Supergraphs of non- DD_2 -graphs

Staying on the algorithmic issue, given a non- DD_2 -graph G , one can ask the following natural question: What is the smallest number of edges which added to G result in a DD_2 -graph? In particular, one can consider the following decision variant of this problem.

Problem 4.1 (*The Min-to- DD_2 Problem*). Let G be a non- DD_2 -graph and let k be a positive integer. Is it possible to add at most k edges to G such that the resulting graph becomes a DD_2 -graph?

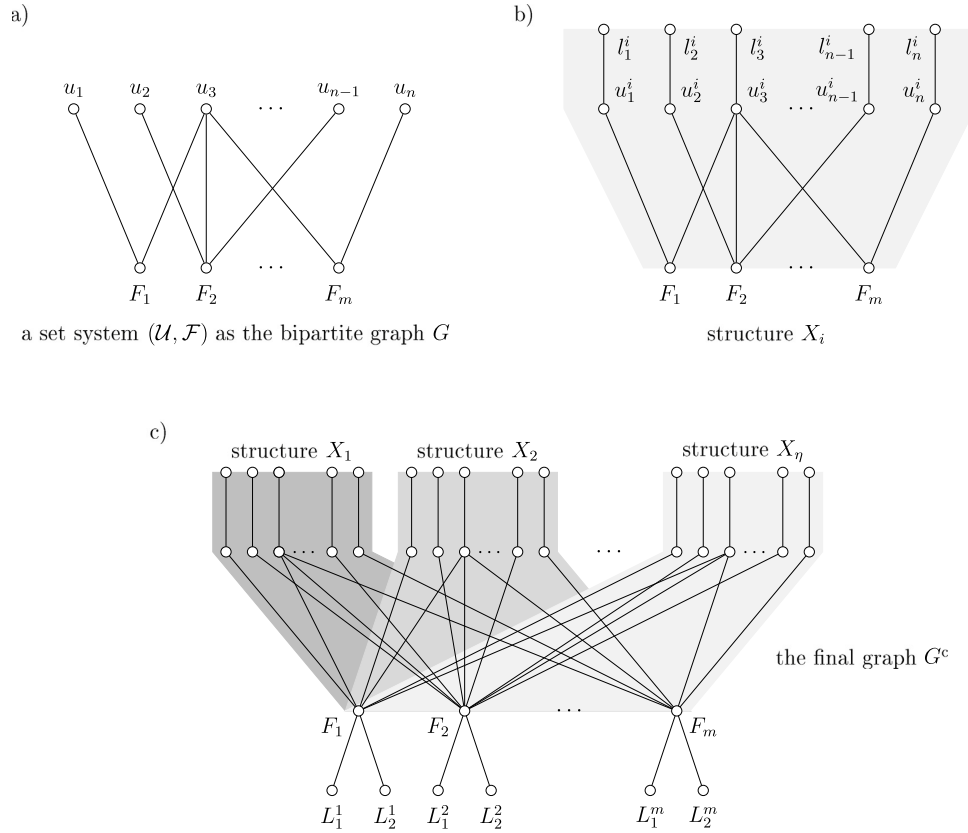


Fig. 3. Reduction from the Set Cover problem.

In this section, by reduction from the Set Cover problem [30,31], we show that the Min-to- DD_2 problem is NP-complete (and so its optimization variant is NP-hard).

Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be a set of n items and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of m sets containing the items in $\mathcal{U}$, i.e., each $F_i \subseteq \mathcal{U}$, such that each element in $\mathcal{U}$ belongs to at least one set from $\mathcal{F}$; we assume that $(\mathcal{U}, \mathcal{F})$ is represented as the bipartite graph $G = (V_G, E_G)$ with the partition $V_G = \mathcal{U} \cup \mathcal{F}$, where $\{u, F\} \in E_G$ if and only if $u \in F$, for $u \in \mathcal{U}$ and $F \in \mathcal{F}$; see Fig. 3(a). A k -element subset of $\mathcal{F}$, whose union is equal to the whole set $\mathcal{U}$, is called a *set cover of size k* .

Problem 4.2 (The Set Cover Problem). Let $(\mathcal{U}, \mathcal{F})$ be a set system, and let $k \leq |\mathcal{F}|$ be a positive integer. Does $(\mathcal{U}, \mathcal{F})$ possess a set cover of size k ?

The Set Cover problem is well known to be NP-complete [31]. We are going to prove that for a given set system $(\mathcal{U}, \mathcal{F})$, represented as the bipartite graph $G = (\mathcal{U} \cup \mathcal{F}, E_G)$, and a positive integer $k \leq |\mathcal{F}|$, there exists a set cover of size k if and only if there is a solution for the Min-to- DD_2 problem in the graph $G^c = (V_{G^c}, E_{G^c})$ (see Fig. 3 for an illustration), with the same parameter k , where:

- $V_{G^c} = \mathcal{F} \cup \bigcup_{i=1}^{\eta} \{u_1^i, \dots, u_n^i\} \cup \bigcup_{i=1}^{\eta} \{l_1^i, \dots, l_n^i\} \cup \bigcup_{j=1}^m \{L_1^j, L_2^j\}$, with $\eta = 2k + 1$;

- $E_{G^c} = \check{E} \cup \bar{E}$, where $\check{E} = \bigcup_{j=1}^m \bigcup_{u_i \in F_j} \{u_1^i F_j, u_2^i F_j, \dots, u_n^i F_j\}$, and $\bar{E} = \bigcup_{i=1}^n \{u_1^i l_1^i, u_2^i l_2^i, \dots, u_n^i l_n^i\} \cup \bigcup_{j=1}^m \{F_j L_1^j, F_j L_2^j\}$.

Notice that $L_{G^c} = \bigcup_{i=1}^n \{l_1^i, l_2^i, \dots, l_n^i\} \cup \bigcup_{j=1}^m \{L_1^j, L_2^j\}$, and $S'_{G^c} = \bigcup_{i=1}^n \{u_1^i, u_2^i, \dots, u_n^i\}$, while $S''_{G^c} = \mathcal{F}$. Moreover, none of weak supports in G^c has a non-leaf non-support neighbor, i.e., we have $N_{G^c}(s) - L_{G^c} = \{F \in \mathcal{F} : s \in F\} \subseteq S''_{G^c}$ for each weak support $s \in S'_{G^c}$, and hence G^c is not a DD_2 -graph by [Theorem 2.4](#). In addition, the set S'_{G^c} of weak supports is independent in G^c . Clearly, our reduction takes polynomial time: the order of G^c is equal to $(2k+1)n + 3m \leq (2m+1)n + 3m$, while its size equals $2m + (2k+1)(n + \sum_{j=1}^m |F_j|) \leq 2m + (2m+1)(n + \sum_{j=1}^m |F_j|)$.

Lemma 4.3. *Let $\langle G = (\mathcal{U} \cup \mathcal{F}, E), k \rangle$ be an instance of the Set Cover problem (with $k \leq |\mathcal{F}|$). Then for $(\mathcal{U}, \mathcal{F})$ there exists a set cover of size k if and only if there exists a solution to the Min-to- DD_2 problem for the instance $\langle G^c, k \rangle$.*

Proof. Without loss of generality assume that $\mathcal{C} = \{F_1, F_2, \dots, F_k\}$ is a solution to the Set Cover problem for $\langle G, k \rangle$. Let G_k^c be the graph resulting from G^c by adding k edges $L_1^1 L_2^1, L_1^2 L_2^2, \dots, L_1^k L_2^k$. Since $\mathcal{C}$ is a set cover of G , we have $N_{G_k^c}(s) - (L_{G_k^c} \cup S_{G_k^c}) \neq \emptyset$ for every weak support $s \in S'_{G_k^c}$, and hence G_k^c is a DD_2 -graph by [Theorem 2.4](#).

On the other hand, let $\mathcal{E} = \{e_1, \dots, e_k\}$ be a solution to the Min-to- DD_2 problem in G^c . If $k = |\mathcal{F}|$, then $\mathcal{F}$ itself constitutes the required set cover for $(\mathcal{U}, \mathcal{F})$. Thus assume $k < |\mathcal{F}|$. Let $U_i = \{u_1^i, \dots, u_n^i\}$ be the subset of weak supports in G^c , $i = 1, \dots, \eta$. In order to be a DD_2 -graph by $G_{\mathcal{E}}^c = G^c + \mathcal{E}$ ($= G_{\mathcal{E}}^c + \{e_1, \dots, e_k\}$), taking into account [Theorem 2.4](#) and the structure of G^c , in particular, $\eta = 2k+1$ and the fact that all its weak supports are pairwise independent, it follows that for each U_i , there exists j_i such that the vertex F_{j_i} is not a support vertex in $G_{\mathcal{E}}^c$ any longer. Let $J = \{j_i : i \in \{1, 2, \dots, \eta\}\}$. Next, observe that to make F_{j_i} a non-support vertex in $G_{\mathcal{E}}^c$, there must be an edge in $\mathcal{E}$ incident to the leaf $L_1^{j_i}$ and an edge in $\mathcal{E}$ incident to the leaf $L_2^{j_i}$.

If $\mathcal{E}$ is *consistent*, that is, if each of the edges in $\mathcal{E}$ is of the form $L_1^j L_2^j$ for some $j \in \{1, 2, \dots, m\}$, then the family $\{F_j : j \in J\}$ constitutes a solution of size at most k to the Set Cover problem for $\langle G, k \rangle$, as $|J| = k$.

Therefore assume that $\mathcal{E}$ is not consistent. The idea is to replace $\mathcal{E}$ with another set of at most k edges which also is a solution to the Min-to- DD_2 -problem for $\langle G^c, k \rangle$, but which is consistent. Our replacement is based upon the following simple claim.

Claim 5. *Let $H = (V_H, E_H)$ be a graph with $\delta(H) \geq 1$. If S is a set of disjoint pairs of vertices in H , then $|S| \leq |E_H|$.*

In particular, considering H as the graph whose edge set is $\mathcal{E}$ and vertex set is the set of all endpoints of edges in $\mathcal{E}$, and setting $S = \bigcup_{j \in J} \{(L_1^j, L_2^j)\}$, we obtain $|J| = |S| \leq k$. Therefore, the set $\mathcal{E}' = \bigcup_{j \in J} \{L_1^j L_2^j\}$ is of size at most k and, by the definition of the set J , it also constitutes a solution to the Min-to- DD_2 problem for $\langle G^c, k \rangle$ (as each edge $L_1^j L_2^j$ makes F_j a non-support vertex in $G^c + \mathcal{E}'$, for each $j \in J$). Since $\mathcal{E}'$ is consistent from the definition, the family $\{F_j : j \in J\}$ is a solution of size at most k to the Set Cover problem for $\langle G, k \rangle$ as $|J| \leq k$. $\square$

A non-deterministic polynomial algorithm for the Min-to- DD_2 problem in G just guesses at most k “missing” edges and checks whether the graph resulting from adding these edges to G is a DD_2 -graph (which can be done in polynomial time by [Remark 1](#)). Hence by [Lemma 4.3](#) and the fact that the Set Cover problem is NP-complete [\[31\]](#), we obtain the following theorem.

Theorem 4.4. *The Min-to- DD_2 problem is NP-complete. Consequently, given a non- DD_2 -graph G , the problem of determining the minimum number of missing edges after adding which the resulting graph becomes a DD_2 -graph is NP-hard.*

Finally, since being a DD_2 -graph is closely related to the operation of subdivision an edge (see [Theorem 2.6](#)), given a non- DD_2 -graph, one can also ask for the minimum number of such operations after applying which the resulting graph becomes a DD_2 -graph. Fortunately, in this case the problem is polynomially tractable, by a simple reduction to the maximum matching problem. Namely, let X_G denote the set of all weak supports of a graph G such that $N_G(s) - (L_G \cup S_G) = \emptyset$ for every $s \in X_G$ (if G is a DD_2 -graph, then X_G is the empty set). Consider now the graph H resulting from subdivision of an edge $e \in E_G$. Observe that $X_H \subseteq X_G$ and $|X_H| \geq |X_G| - 2$, in other words, subdividing e excludes at most two vertices in X_G . In particular, $|X_H| = |X_G| - 2$ if and only if $e = s_1s_2$ for some $s_1, s_2 \in X_G$. Consequently, the minimum number of edge subdivisions is equal to $|X_G| - |M|$, where M is the maximum matching in the induced subgraph $G[X_G]$, which – taking into account [\[33\]](#) – immediately results in the following theorem.

Theorem 4.5. *Given a non- DD_2 -graph G , the minimum number of edge subdivision in G after applying which the resulting graph becomes a DD_2 -graph can be computed in $O(|V_G|^2|E_G|)$ time.*

5. Closing open problems

We close this paper with the following list of open problems that we have yet to settle.

Problem 5.1. Characterize the class of spanning supergraphs of minimal DD_2 -graphs G for which $\gamma\gamma_2(G) = |V_G|$, see [Remark 4](#).

Problem 5.2. Characterize the class of spanning supergraphs of minimal DD_2 -graphs for which the smallest dominating set is a certified dominating set, see the definition before [Theorem 2.4](#).

There are also some algorithmic issues related to the NP-hard optimization problems discussed in Sections 3 and 4. For example, observe that by adding edges connecting distinct pairs of leaves that are adjacent to weak supports, we obtain a DD_2 -graph. However, for the corona graph G of a star $K_{1,n}$ of order at least three – notice that such G is not a DD_2 -graph – it is enough to add only one properly chosen edge of this type. Therefore, we state the following three problems.

Problem 5.3. Provide an efficient approximation algorithm for the problem of determining:

- (1) a spanning minimal DD_2 -graph of the minimum size;
- (2) a spanning minimal DD_2 -graph of the maximum size;
- (3) the smallest number of edges which added to a graph result in a DD_2 -graph.

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OPEN Theoretical modelling of efficient fire safety water networks by certified domination

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This paper explores a new way of designing water supply networks for fire safety using ideas from graph theory, focusing on a method called certified domination. Ensuring a good water supply is crucial for fire safety in communities, this study looks at the rules and problems in Poland for how much water is needed to fight fires in different areas and how this can be achieved at a lowest possible cost. We present a way to plan water supply networks for fire protection as a graph, where each point (node) is a place that needs water, and the lines (links) show where water can go between these points. The main idea is to find the best places to put pumping stations and wells in the network to save money and still meet all the fire safety requirements. Our approach assumes that it costs more to build a pumping station than a well. We use some examples to show how this method can find cost-effective solutions for water supply networks, while ensuring that they meet fire safety requirements and are not too expensive to build. This approach is a new and efficient way to improve the design of water supply networks for fire safety. Key challenges that are solved in this paper are a linear time algorithm finding an optimal solution for networks without cycles and a BLP (Binary Linear Programming) algorithm solving the problem in arbitrary networks.

Keywords Fire protection, Water supply networks, Graph theory, Certified domination

Providing water to the fire protection water supply network is a crucial aspect of the overall fire protection and life safety strategy of an entire community. To meet the expectations of fire safety when a new building is developed or an existing building is renovated, necessary calculations are performed so that the building is complied with fire safety regulations. Before everything it is important to make sure that the proper amount of water is available to the responding fire department for both suppression of the fire in the building, and protection of any exposed buildings. All water-based fire protection systems need water. Without access to an adequate water supply these systems will not function properly. When determining a water supply one needs to make sure it is reliable, and has sufficient volume and pressure to meet the system demand. Because of these reasons, the fire safety regulations require a minimum amount of water be provided, as well as appropriate water pressure, network construction, and supply diameters for external hydrants or automatic sprinkler systems.

In Poland the provision of fire water supply for external fire extinguishing is required for all settlement units with a population exceeding 100 people, not constituting colony buildings, and within their boundaries: public utility buildings and collective residences, as well as production and storage buildings¹. The demand of the amount of water may be decreased in rural areas with no buildings with increased fire safety, while, for example, in a production and warehouse building the required amount of water for fire-fighting purposes for external fire extinguishing should be higher.

The fire protection water supply network should be constructed as a circumferential network. It is allowed to build a branched fire-fighting water supply network outside urban areas. Moreover, it is allowed to build branches from the peripheral network to supply external hydrants.

In Poland the fire protection network can be a part of the main supply network, and in general they are not separated. If in a settlement unit the water resources intended for the population supplied via a water pipeline do not provide the amounts required for fire protection purposes, at a distance of no more than 250 m from the outermost buildings of the settlement unit or the protected building, at least one of the following supplementary water sources with sufficient parameters is provided:

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- a well;
- a water intake point at a natural or artificial water reservoir;
- fire-fighting water tank meeting the requirements of the Polish Standards².

The supplementary water sources are used in places with no water supplied via a water pipeline or when the supplied pipeline water has insufficient parameters (capacity, pressure, efficiency). This solution is applied, if necessary, for example, to hotels or sport facilities in rural areas, usually where the fire-fighting water supply network is branched¹.

The supply system in the water supply network includes the elements such as water intakes, first stage pumping stations (pumping stations from the intake to the treatment station), water treatment plants (if needed), clean treated water tanks, second stage pumping stations (pumping stations from the reservoir to the network) and network expansion tanks³.

Act on collective water supply and collective sewage disposal provides that collective water supply and collective sewage disposal is the commune's own responsibility, but also municipalities are not obliged to build water supply and sewage facilities directly to every object. Hence, while the local community office should provide fire water supply with the required parameters in the neighbourhood of each newly built (or rebuilt) object, in practice it may take a few years (due to lack of funds) and in such situations investors usually prefer to build wells, water tanks or use water from a nearby lake (if there is any) to fulfill the fire safety regulations, however by their own expense. Our model is meant to help develop planning techniques which can be applied in fulfilling the fire safety regulations at a lowest possible cost at the same time.

In this paper we introduce a theoretical model of a water supply network given in the language of graph theory. The model focuses on placing the water supply issues and hence other, less important parameters are omitted. The we apply the notion of minimum certified domination sets to solve placing water supplies: wells and pumping stations in such a way their number for the entire network is minimized. Other types of domination numbers do not fulfill all the requirements of our model, specifically do not take into account placing two different facilities, so they are not suitable in this model.

Results

This section introduces definitions and properties of certified domination in graphs. Afterwards we study computational results concerning finding a minimum certified dominating sets. In particular, we give some basic results obtained in previous works to help understand the background for our study. Then we present a linear time algorithm finding a minimum certified dominating sets in trees. A tree is a connected graph in which any two node are connected by exactly one path. Next we present a binary integer programming (BLP) formulation that finds a minimum certified dominating sets in arbitrary graphs. Even though the BLP works for any graph, its computational complexity is greater. Performances of both algorithms are compared in next chapters.

Graph theory definitions and network model

Let $G = (V, E)$ be a model of a water supply network where V is a set of nodes, where each node represents a place that needs water, and E is the set of links between nodes. Two places u and v are connected if and only if placing a supply system in u gives water in the required quantity and parameters at v . This may be influenced by the landforms, natural resources (e.g. nature reserves, forests), type of developments in the area and other factors. We assume that if u is connected to v then v is also connected to u , and thus G is an undirected graph (see Fig. 4). It is worth to note that the network is more dense in urban areas, as in this case town Liniewo. In rural areas nodes have lower degree, and nodes of higher degree are more distant from each other.

In what follows we assume, without loss of generality, that the only possible supplementary water source (apart from pumping station) is a well. . Our task is to place in G pumping stations and individual wells in such a way that the total number of water supply devices is as small as possible providing the following conditions are met:

1. **Connection to Pumping Stations** Each node without a pumping station and without a well must be connected to a node with a pumping station. This ensures that every location receives adequate water supply from a nearby pumping station.
2. **Pressure Maintenance for Wells** If a node has a well, every node connected to it should be equipped either with a well or a pumping station. This prevents a drop in water pressure and ensures that the parameters at the node with the well remain within acceptable limits.
3. **Financial Viability of Pumping Stations** The construction of a pumping station is financially viable only in locations adjacent to at least two other locations without wells and without pumping stations. This criterion ensures that pumping stations are built in locations where they provide maximum benefit and cost-effectiveness.

Our aim is to minimize the total number of wells and pumping stations.



Figure 1. Graphs P_1 , P_2 and P_3 .



Figure 2. Graph P_4 .

For example, (see Fig. 1) if the water supply network is just one node, P_1 , then it is enough to build there a well. If the network are two adjacent nodes, namely P_2 , then the optimal solution is to build wells in both nodes. For P_3 the optimal solution is to build a pumping station in the middle node.

However the most interesting case among small graphs is P_4 , see Fig. 2.

Note that placing a pumping station in x_1 or in x_4 is not possible by condition 3. Similarly, placing one pumping station (for example at node x_2) and one well (at x_4) is impossible by condition 2. Hence, the optimal solution for P_4 is to build a well at each node.

For the graph G_1 (see Fig. 3, left), the optimal solution is one pumping station and two wells, while for the graph G_2 (see Fig. 3, right), which has no support vertices, and all nodes have a minimum degree of 2, the optimal solution is to place two pumping stations.

We generally follow the notation and terminology of a book on domination in graphs⁴. For a graph G , the set of nodes is denoted by V and the links set by E . For a vertex $v \in V$, the *open neighbourhood* $N(v)$ of v is the set of all nodes adjacent to v , and $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v . The *degree* of a node v in G is $d_G(v) = |N(v)|$. The number $\min\{d_G(v) : v \in V\}$ is the minimum degree of G and is denoted by $\delta(G)$. A node of degree 0 is called an *isolated vertex*, while a node of degree one in G is called a *leaf* of G . If v is a leaf, then its only neighbour is called a *support vertex* of v . If a support vertex is adjacent to exactly one leaf, then we call it a *weak support vertex* and a *strong support vertex*, otherwise.

Given a graph G , we say that a subset $D \subseteq V$ is a *dominating set* of G if every node belonging to $V - D$ is adjacent to at least one node in D . The *domination number* of a graph G , denoted by $\gamma(G)$, is the cardinality of a smallest (that is minimum) dominating set of G . A dominating set of G of minimum cardinality is called a γ -set of G .

A subset $D \subseteq V$ is called a *certified dominating set* of G if D is a dominating set of G and every node belonging to D has either zero or at least two neighbours in $V - D$. The cardinality of a smallest (that is minimum) certified dominating set of G is called the *certified domination number* of G and is denoted by $\gamma_{\text{cer}}(G)$. A certified dominating set of G of minimum cardinality is called a γ_{cer} -set of G . The certified domination was introduced by Dettlaff et al.⁵ and continued in⁶ in order to describe some possible relations in social networks.

Let $G = (V, E)$ be a model of a water supply network and let D be a set of all nodes in which a pumping station or a well is placed in such a way all requirements 1.–3. are met. Then D is a dominating set of G , because of condition 1. Moreover, if the total number of pumping stations and wells in G is as low as possible, then D is a minimum certified dominating set by conditions 2. and 3.

On the other hand, let $D \subseteq V$ be a minimum certified dominating set of G . If $v \in D$ and v is adjacent to at least 2 nodes not in D , then in the optimal solution we should place a pumping station in v . If $v \in D$ is not adjacent to any nodes not in D , then we should place a well in v .

Figure 4 shows a case study of placing wells and pumping stations at a lowest possible number while meeting conditions 1.–3. Wells are denoted in blue, while pumping station – in red. Every place, denoted black, has a well or a pumping station, or is adjacent to a place with a pumping station, so condition 1. is met. Additionally, if a place has a well, each neighbour also has a well or a pumping station (condition 2.) Also, every place with a pumping station is adjacent to at least two places with no water source. Since the network is more dense in urban areas, the optimal solution suggests building wells only in the suburbs and rural areas.

It is observed⁵ that every support vertex in a graph G is part of every certified dominating set of G . For connected graphs of order at least three, there is an interesting equivalence between the standard domination number and the certified domination number. This equivalence exists when the graph has a γ -set where each vertex has at least two neighbours outside the set. Furthermore, for any graph with a minimum degree of at least two, it is found that there's always a γ -set where every vertex satisfies the neighbourhood condition, leading to equal domination numbers. The results also show that graphs with a unique γ -set naturally have equal domination and certified domination numbers. For connected P_4 -free graphs, except K_2 , these domination numbers are also equal.

Background and motivation

In graph theory, the concept of a certified domination set has received considerable attention. Many scholarly articles explore different aspects of this subject. The idea of certified domination has been explained using different practical examples⁵. In this paper the researches have calculated exact values for the certified domination number in elementary graph types and set basic upper limits for this number in any type of graph. They also



Figure 3. Graph G_1 (left) and graph G_2 (right). PS denotes a pumping station, while W denotes a well.

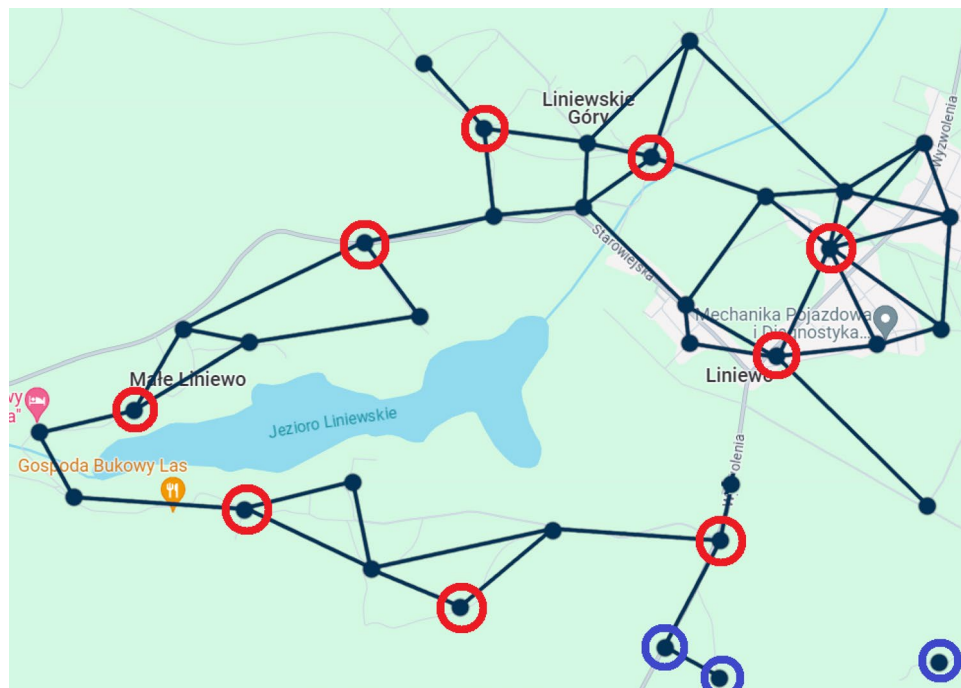


Figure 4. Map with an optimal solution for placing wells (blue circles) and pumping stations (red circles). Source of the map: Google Maps.

noticed that the certified domination number is equal to the well studied ordinary domination number in graph without weak support vertices. On the other hand, the authors in⁷ have proven that the question if the certified domination number is equal to the domination number is NP-complete even for graphs with just one weak support vertex. For this reason using known algorithms for finding the domination number does not solve the problem of the minimum certified dominating set. At the same time studying the certified domination number imposes new challenges requiring new techniques, tools and algorithms.

The motivation behind this paper also comes from the complexity of the certified dominating set problem, which is known to be NP-hard even for split graphs, star convex bipartite graphs, comb convex bipartite graphs, and planar graphs⁸. Therefore an important open problem in this area is ascertaining the computational complexity of identifying the certified domination number across various graph classes. Despite so many studies, the challenge of finding the certified domination number for non-elementary graph classes remains unresolved. Our work aims to address this gap by providing a solution to the open problem presented in⁶ and⁹, specifically targeting the certified domination number for trees (connected acyclic graphs). Despite the relative simplicity of finding dominating sets in trees¹⁰, the task of finding certified dominating sets in trees is significantly more difficult due to the certification requirement. This additional complexity renders the problem both challenging and intriguing for theoretical research and practical use in graph theory. The linear time algorithm presented in this paper finds a minimum certified dominating set in trees, while the ILP algorithm finds a minimum certified dominating set for any graph, however in exponential time. Other studies have compared traditional and certified domination number in graphs⁶, and have analyzed certified perfect dominating sets¹¹. Research on isolate-free graphs has explored the structural properties of graphs related to certified domination¹². The concept of certified critical vertices and vertex certified domination critical graphs were also examined⁹.

Mathematical models and tools has been used for solving various problems in water distribution networks since many years, see a review article¹³. Depending on the problem to solve, the mathematical methods include for example global gradient method, perturbation or delta expansion, Newton-Raphson methods, linear methods¹⁴, extended linear graph theory^{15,16}, genetic algorithms¹⁷, and many others.

Linear algorithm for trees

The tree-order is the partial ordering on the vertices of a tree with $u < v$ if and only if the unique path from the root to v passes through u , see a tree in Fig. 5. Let $C(v)$ be the set of all children of a vertex v , and $F(v)$ the father of v (if v is the root, then $F(v) = \text{null}$). Denote by $S(T)$ the set of all support nodes and by $SW(T)$ the set of all weak support nodes of T .

The algorithm described in this section proceeds the nodes of T along its tree ordering, from the last to the root. It assumes that the number of nodes of tree, denoted $n(T)$, is greater than 2, and the root is not a leaf. The algorithm consists of three phases, Phase 0, Phase 1 and Phase 2.

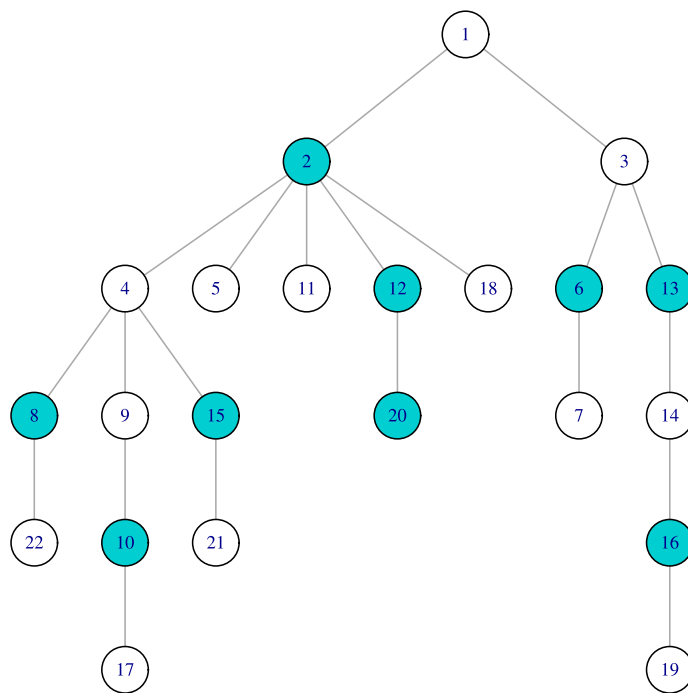


Figure 5. Rooted tree T_1 . Blue nodes form the minimum certified dominating set.

Data: A tree T with tree-order
Result: T with identified L , SW and S nodes

```

1 Function Phase0 of MinimumCertifiedDominatingSet ( $T$ ):
2   for  $v \in V(T)$  (from the last to the root) do
3     if  $v$  is of degree 1 then
4        $v.sta \leftarrow L$ ;
5        $F(v).leaf \leftarrow F(v).leaf + 1$ ;
6        $F(v).add \leftarrow v$ ;
7     else
8       if  $v.leaf > 0$  then
9         if  $v.leaf = 1$  then  $v.sta \leftarrow SW$  else  $v.sta \leftarrow S$ ;
10    return  $T$ 

```

Algorithm 1. Phase 0 of MinimumCertifiedDominatingSet

In Phase 0 of MinimumCertifiedDominatingSet (Algorithm 1), or MCDS for short, determines in T : leaves (assigns to such nodes status L), weak support vertices (assigns to such nodes status SW) and strong support vertices (status S). For this reason, each node stores local variables, sta , $leaf$ and add and their values are updated while performing the algorithm. Before the algorithm starts, we assume each variable of every node is equal to 0. To determine support nodes, the algorithm analyzes values of the local variables $leaf$. Moreover, every weak support should remember his unique leaf (variable add , it is used further by SW nodes only). The Phase 0 Algorithm works in linear time.

Data: A tree T with tree-order and identified L , SW and S nodes

Result: T with determined statuses

```

1 Function Phase1 of MinimumCertifiedDominatingSet ( $T$ ):
2   for  $v \in V(T) - L(T)$  (from the last to the root) do
3     if  $v.leaf > 0$  then
4       if  $v.n_0 + v.n_1 + v.n_{22} = 0$  and  $v.leaf = 1$  then  $F(v).sw \leftarrow F(v).sw + 1$ ;
5       else  $v.sta \leftarrow S$ ;
6        $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
7     else
8       if  $v.n_0 = 0$  then
9         if  $v.n_2 > 0$  then
10            $v.sta \leftarrow 1$ ;
11            $F(v).n_1 \leftarrow F(v).n_1 + 1$ ;
12         else
13            $v.sta \leftarrow 0$ ;
14            $F(v).n_0 \leftarrow F(v).n_0 + 1$ ;
15       else
16         if  $v.n_0 = 1$  and  $v.n_1 + v.sw = 0$  then
17            $v.sta \leftarrow 22$ ;
18            $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
19            $F(v).n_{22} \leftarrow F(v).n_{22} + 1$ ;
20         else
21           if  $F(v).sta = SW$  then  $f = 1$  else  $f = 0$ ;
22           if  $v.n_0 > v.sw + f$  then
23              $v.sta \leftarrow 2$ ;
24              $F(v).n_2 \leftarrow F(v).n_2 + 1$ ;
25           else
26              $v.sta \leftarrow 11$ ;
27              $F(v).n_1 \leftarrow F(v).n_1 + 1$ ;
28   if  $root.n_2 = 0$  then  $root.sta \leftarrow 2$ ;
29   return  $T$ 

```

Algorithm 2. Phase 1 of MinimumCertifiedDominatingSet

At this point we assume that all additional local variables (namely $n_0, n_1, n_2, n_{22}, sw$) are equal 0 at the beginning of the performance of the Algorithm 2 Phase 1, while $sta, leaf, add$ have values as determined in Algorithm 1 Phase 0. At Phase 1 sta may also be equal 0, 1, 11, 2 or 22. In general, when v is dominated by its child, $v.sta = 1$, while $v.sta \in \{2, 22\}$ means that v should be a member of a minimum certified dominating set of T , together with support vertices (statuses SW or S). Statuses 11 and 22 have special meaning, explained later. In some cases there is no need to change the status of a node from 0 to 1, so its status remains unchanged. It means that at the end of the performance of the algorithm nodes with $sta = 0$ or $sta = 1$ are certified dominated by the nodes with $sta \in \{2, 22, S, SW\}$. After performing the main loop (lines 2–27) on node v , the local variable n_0 shows how many children of v have status 0, n_1 is equal to the number of children of v with $sta \in \{1, 11\}$, n_2 is equal to the number of children of v with $sta \in \{2, 22, S, SW\}$ and sw is equal to the number of children of v with status sw .

Let $L(T)$ be the set of all nodes of status L , that is leaves. Algorithm 2 Phase 1 proceeds only non-leaf nodes (line 2). The statements differ depending whether node v is a support node (then lines 4–6 are performed) or not (lines 8–27 are performed). Line 28 is performed additionally for the root node only.

If v is a support vertex, then the condition (line 4) checks whether v has exactly one leaf child and no other child with status 0 or status 1 or status 22. If that is the case, the father of v gets to know that there is one more node with these properties. If v is a weak support vertex but has at least one non-leaf child with status 0 or status 1 or status 22, then there is no risk at the end of the Phase 2 that v has exactly one neighbour not in the minimum certified dominating set of T . Hence, v may be treated as a strong support vertex and therefore its status is changed to S . In line 6 we increase the local variable n_2 of the father of v by one.

Lines 8–27 are performed for nodes, which are neither leaves nor support vertices. If v does not have children that should be dominated by v (with status 0) and has a child with $sta \in \{2, 22, S, SW\}$, then it simply gets status 1 (lines 8–11). If v neither has children with status 0 nor status 2, then it gets status 0.

Lines 16–27 are performed for nodes with at least one child with status 0. Statuses 11 and 22 indicate that additional actions may be taken on those nodes and its children or parents during Phase 2.

Since there is only one **for** loop, the algorithm MinimumCertifiedDominatingSet in Phase 1 works in linear time.

Data: A tree T with determined statuses
Result: T with determined minimum certified dominating set

```

1 Function Phase2 ( $T$ ) :
2   for  $v \in V(T) - L(T)$  (from last to the root) do
3     if  $v$  is not the root then
4       if  $v.sta = SW$  and  $F(v).sta \in \{2, SW, S\}$  then  $(v.add).sta \leftarrow 2$ ;
5       if  $v.sta = 0$  and  $F(v).sta = 11$  then  $v.sta \leftarrow 2$ ;
6       if  $v.sta = 0$  and  $F(v).sta = 22$  and  $F(F(v)).sta \in \{2, 22, SW, S\}$  then
7          $v.sta \leftarrow 2$ ;
8          $F(v).sta \leftarrow 1$ ;
9         if  $F(F(v)).sta = SW$  then  $F(F(v)).sta = S$ ;
10    if  $root.sta = SW$  and  $root.n_0 + root.n_1 = 0$  then  $(root.add).sta \leftarrow 2$ ;
11     $D \leftarrow \{v \in V(G) : v.sta \in \{2, 22, S, SW\}\}$ ;
12    return  $D$ 

```

Algorithm 3. Phase 2 of MinimumCertifiedDominatingSet

The algorithm MinimumCertifiedDominatingSet in Phase 2 deals with possible situations where a node may dominate exactly one node outside the minimum certified dominating set D . This may happen when v is a weak support vertex and its father is supposed to belong to D (lines 4–5). Then the child of v is added to D . For similar reasons line 6 is performed for particular situation, as well as lines 7–10.

At the end, the minimum certified dominating set D consisting of nodes with status 2, 22, S or SW is returned.

Since there is only one for loop, the algorithm MinimumCertifiedDominatingSet in Phase 2 works in linear time and therefore, the complete algorithm works in linear time.

Figure 5 shows results of performing the MCDS algorithm on trees. Nodes in blue (set D) show places to build a pumping station or a well. Specifically, nodes 2, 6, 8, 10, 13, 15 and 16 should be placed pumping stations, while nodes 12 and 20 have no neighbours in $V - D$, so in those places wells should be built.

Correctness proof of the linear algorithm

Denote by T_v the tree induced by $v \in V(T)$ and all its descendants. For example, T_9 for the tree T in Fig. 5 is the path (9, 10, 17).

The height of a tree (also known as depth), denoted $h(T)$, is the maximum distance between the root node of the tree and the leaf node of the tree.

Theorem 1 The MinimumCertifiedDominatingSet (MCDS) performed on a rooted tree T , $n(T) > 2$, where the root is not a leaf, finds a minimum certified dominating set of T .

Proof If $h(T) = 1$, then T is a star with at least two leaves (because the root is not a leaf). In Phase 0 the root gets status S and every other node status L. In Phase 1 only one thing happens: the root changes status to 2. In Phase 2 the set D consists of 1 element, namely the central node of the star, which is the minimum certified dominating set of this tree.

In what follows we proceed by the induction on the number of nodes in a tree T . Assume that for every tree T' with $n(T') < n(T)$ the MCDS algorithm finds a minimum certified dominating set of T' and assume $h(T) > 1$.

Let $v_0, v_1, \dots, r$ be the longest path from a leaf to the root r . Then v_0 is a leaf and v_1 is a support vertex. If additionally v_2 is a support vertex, then at the end of the algorithm MinimumCertifiedDominatingSet (MCDS) performed on T_{v_1} , $v_2 \in D$ and, if v_1 is a weak support vertex, then $v_0 \in D$. Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$ if v_1 is a strong support vertex or $\gamma_{cer}(T) = \gamma_{cer}(T') + 2$ if v_1 is a weak support vertex. By the hypothesis, the algorithm MCDS finds the minimum certified dominating set of T' . Hence MCDS also finds the minimum certified dominating set of T .

Assume now that v_2 is not a support vertex and v_2 has at least two children that are support vertices. Then at the end of the algorithm MCDS performed on T , $v_2 \notin D$ (in Phase 1 lines 8–11 gets status 1), every child of v_2 belongs to D and each such child has at least two neighbours outside D . Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$, regardless whether v_1 is a strong support vertex or a weak support vertex. The rest of the proof follows as in previous case.

In what follows we assume that v_2 has exactly one child, namely v_1 . Then, since the root is not a leaf, $h(T) > 2$. Moreover, if x_0 is a leaf such that the distance from r to x_0 is the same as the distance from r to v_0 , then $(x_0, x_1, x_2, \dots, r)$ is a path from x_0 to r , then by applying a similar reasoning as in previous paragraph, we obtain that $\deg(x_1) = \deg(x_2) = 2$, so this we assume for the remaining part of the proof.

If v_3 is a support vertex, then at the end of the algorithm MCDS performed on T , $v_0, v_2 \notin D$ and $v_1, v_3 \in D$ and v_3 has at least two neighbours outside D (at least one leaf and v_2). Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$. The rest of the proof follows as in previous cases.

Now assume that v_3 is not a support vertex, but has at least one child which is a support vertex. Denote such a child by z . Then by our assumptions, every child of z is a leaf. In this situation at the end of performing the algorithm MCDS on T , $v_0, v_2, v_3 \notin D$ and $v_1, z \in D$ and z has at least two neighbours outside D (at least one leaf and v_3). Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$. If v_3 is not a leaf in T' then the rest of the proof follows

as in previous cases. If v_3 is a leaf in T' , then v_3 is a root, T' is a star. It is easy to verify that in this situation the algorithm returns D consisting of the two support vertices of P_6 , namely $D = \{v_1, z\}$.

Assume that every child of v_3 is a father of exactly one support vertex. Then by applying the algorithm to T , every leaf in T_{v_3} gets status 0, every support vertex gets status SW and every child of v_3 gets status 1. If $v_3 = r$, then at the end of Phase 1 v_3 gets status 2 and it is easy to see that D is a minimum certified dominating set of T . If $v_3 \neq r$, then by applying the algorithm to T , v_3 gets status 0 in Phase 1. Depending on the statuses of other children of v_4 , if v_4 gets status 11 in Phase 2, or if v_4 gets status 22 and $sta(v_5) \in \{2, 22, SW, S\}$, then at the end of the algorithm $v_3 \in D$. In both cases $v_4 \notin D$ (see lines 6–10 of Phase 3). Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$ and the rest of the proof is similar to the previous cases. Therefore assume $v_3 \notin D$. Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$ and the rest of the proof follows as in previous cases regardless whether v_3 is a leaf in T' or not.

In what follows we assume that v_3 has exactly one child, namely v_2 . Then, since the root is not a leaf, $h(T) > 3$. Assume also that if $(x_0, x_1, \dots, r)$ is another path from a leaf x_0 to r of length $h(T)$, then $deg(x_1) = deg(x_2) = deg(x_3) = 2$.

If v_4 is a support vertex, then at the end of the algorithm MCDS performed on T , $v_0, v_2, v_3 \notin D$ and $v_1, v_4 \in D$ and v_4 has at least two neighbours outside D (at least one leaf and v_3). Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$. The rest of the proof follows as in previous cases.

Now assume that v_4 is not a support vertex, but has at least one child which is a weak support vertex. Denote such children by $z_1, z_2, \dots, z_k$. Then during Phase 1 lines 21–27 are performed for v_4 . If v_4 has more children of status 0 (like v_3) than nodes with status SW in its neighbourhood (which in our case means that v_4 has at least two children of status 0), then v_4 gets status 2 and at the end of performing the algorithm MCDS on T , $v_4, z_1, \dots, z_k \in D$ together with the leaf children of $z_1, \dots, z_k$ (the children are added to D during Phase 2 line 4). Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$ and v_4 has at least two neighbours outside D . Now the rest of the proof follows as in previous cases.

If v_4 does not have more children of status 0 than nodes with status SW in its neighbourhood, then v_4 gets status 11 (lines 25–27) and during Phase 2 v_3 together with every child of v_4 of status 0 get status 2. Hence $v_1, v_3 \in D$ and $v_2, v_4 \notin D$ and each $z_1, \dots, z_k$ has at least two neighbours outside D (one leaf and v_4). Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$, since v_3 has at least two neighbours outside D : v_2 and v_4 . Now the rest of the proof follows as in previous cases (note that T' has one more child with status SW compared to T , so during performing the algorithm on T' node v_4 also gets status 11).

Now assume every child of v_4 , except for v_3 is a strong support vertex. Then in Phase 1 of the algorithm applied for T , v_4 gets status 22 (lines 16–19). Then, if $v_4 \neq r$ and $sta(v_5) \in \{2, 22, SW, S\}$, v_3 gets status 2 during Phase 2 and define $T' = T - T_{v_1}$. If not, v_3 gets status 0 and define $T' = T - T_{v_2}$. In both cases $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$ and the rest of the proof follows as in previous cases. In what follows assume that v_4 is not a support vertex nor has a child which is a weak support vertex nor each child of v_4 is a strong support vertex. Hence, if $deg(v_4) > 2$, then v_4 is a father of a strong support vertex (but not exclusively) or is a grandfather of a support vertex or is a father of at least two children which are grandfathers of support vertices (like v_3). In this case during performing Phase 1 on node v_4 the condition in line 16 is false and condition in line 22 is true, so v_4 gets status 2. Then $v_4 \in D$ and v_4 has at least two neighbours outside D . Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$, since v_4 is a support vertex in T' . Now the rest of the proof follows as in previous cases.

In what follows we assume that v_4 has exactly one child, namely v_3 . Then, since the root is not a leaf, $h(T) > 4$. Then during performing the Phase 2 on v_4 , the condition in line 16 of Phase 1 is true, so v_4 gets status 22. Depending on the status of v_5 either v_3 gets status 2 at the end of Phase 2 or not. If the father of v_4 is of status in $\{2, 22, SW, S\}$ (Phase 2 lines 6–9), then $v_1, v_3 \in D$ and $v_0, v_2, v_4 \notin D$. Define $T' = T - T_{v_1}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$, since v_3 is a support vertex in T' . Now the rest of the proof follows as in previous cases. If the father of v_4 is not of status in $\{2, 22, SW, S\}$, then $v_1, v_4 \in D$ and $v_0, v_2, v_3 \notin D$. Define $T' = T - T_{v_2}$. Then $\gamma_{cer}(T) = \gamma_{cer}(T') + 1$, since v_4 is a support vertex in T' . Now the rest of the proof follows as in previous cases.

This completes the proof. $\square$

Integer linear programming formulation for minimum certified dominating sets

In this section, we present an integer linear programming (ILP) formulation aimed at solving the problem of identifying a minimum certified dominating set D_{cer} in a given graph $G = (V, E)$, where $V = \{i : i = 1, 2, \dots, n\}$ and $|V| \geq 2$. Linear programming is a mathematical optimization technique that involves the minimization or maximization of a linear objective function subject to a set of linear equality or inequality constraints. The objective function in our formulation seeks to minimize the cardinality of a certified dominating set D , while the constraints encapsulate the specific conditions that define a certified dominating set. We provide a theorem followed by a proof of correctness to validate the efficacy of this BLP formulation. It should be noted there is an alternative solution approach discussed in⁸. However, must be emphasized that the formula presented in that article is incorrect, demonstrated by the counterexample of a path P_4 : an optimal feasible solution for this graph is not a certified dominating set. This underlines the importance of the careful proof of correctness presented in this paper. This ILP-based solution provides a mathematical and computationally efficient way to solve this complex, combinatorial problem.

We define the following sets of binary decision variables:

$x_i = 1$ if and only if $i \in D$,

a_i is an auxiliary variable.

The above decision variables determine the vertex set $D = \{i \in V : x_i = 1\}$ of cardinality $|D| = \sum_{i \in V} x_i$.

The linear program to find a minimum certified dominating set D_{cer} in a graph $G = (V, E)$ is formulated as follows:

$$\text{Minimize } Z = \sum_{i=1}^n x_i$$

subject to

$$\forall i \in V, \quad x_i + \sum_{j \in N(i)} x_j \geq 1 \quad (C1)$$

$$\forall i \in V, \quad \sum_{j \in N(i)} (1 - x_j) + |V| \cdot a_i \geq 2x_i \quad (C2)$$

$$\forall i \in V, \quad \sum_{j \in N(i)} (1 - x_j) + |V|(a_i - 1) + 2 \leq 2x_i \quad (C3)$$

$$\forall i \in V, \quad x_i \in \{0, 1\}, \quad a_i \in \{0, 1\} \quad (C4)$$

We present a theorem to establish the validity and correctness of the linear program formulated to find the minimum certified dominating set in a given graph $G = (V_G, E_G)$. The theorem aims to rigorously prove that the linear program not only satisfies the constraints defining a certified dominating set but also optimally minimizes its cardinality.

Let Ω denote the domain determined by all feasible solutions to Z , i.e.

$$\Omega = \{(x, a) : x \in \{0, 1\}^n, a \in \{0, 1\}^n \text{ satisfy (C1), (C2) and (C3)}\}.$$

Proposition 2 Any feasible solution $(x, a) \in \Omega$ determines a certified dominating set.

Proof Let $(x, a) \in \Omega$. Consider $D = \{i \in V : x_i = 1\}$.

- Since (x, a) is feasible, constraint (C1) ensures that every vertex or one of its neighbours is in D , thus satisfying the dominating set condition.
- Constraints (C2) and (C3) work together to satisfy the certified condition: zero neighbours or at least two neighbours in $V - D$ for each vertex in D . Assume first that $x_i = 0$. Then by (C1) $\sum_{j \in N(i)} x_j \geq 1$, and so

$$0 \leq \sum_{j \in N(i)} (1 - x_j) \leq |V| - 2. \quad (1)$$

In this case (C2) simplifies to $\sum_{j \in N(i)} (1 - x_j) + |V| \cdot a_i \geq 0$ which is always true, regardless of a_i . At the same time (C3) simplifies to

$$\sum_{j \in N(i)} (1 - x_j) + |V|(a_i - 1) + 2 \leq 0. \quad (2)$$

By (1) and by substituting $a_i = 0$ in (C3) we obtain $\sum_{j \in N(i)} (1 - x_j) - |V|(a_i - 1) + 2 \leq |V| - 2 - |V| + 2 \leq 0$, so (2) is true. Therefore if (x, a) is a feasible solution and if $x_i = 0$, then substituting $a_i = 0$ the constraints (C2) and (C3) are always true. Assume now $x_i = 1$. Then (C2) simplifies to

$$\sum_{j \in N(i)} (1 - x_j) + |V| \cdot a_i \geq 2, \quad (3)$$

while (C3) simplifies to

$$\sum_{j \in N(i)} (1 - x_j) + |V| \cdot a_i \leq |V|. \quad (4)$$

If $\sum_{j \in N(i)} (1 - x_j) = 0$, then substituting $a_i = 1$ makes (3) and (4) both true. Similarly, if $\sum_{j \in N(i)} (1 - x_j) \geq 2$, then substituting $a_i = 0$ makes (3) and (4) both true. However if $\sum_{j \in N(i)} (1 - x_j) = 1$, then neither $a_i = 1$ nor $a_i = 0$ gives a feasible solution to both (C2) and (C3). Therefore, if (x, a) is a feasible solution and if $x_i = 1$, then (C2) and (C3) are true only when $\sum_{j \in N(i)} (1 - x_j) \neq 1$, which means i does not have exactly one neighbour in $V - D$. Hence if (x, a) is a feasible solution, then D is a certified set.

The conclusion of the above analysis is that any feasible solution $(x, a) \in \Omega$ determines a certified dominating set $D = \{i \in V : x_i = 1\}$. $\square$

Below we see that the reverse of the above result also holds. In particular

Proposition 3 Any certified dominating set D can be associated with a solution $(x, a) \in \Omega$.

Proof Let D be a certified dominating set of G . Consider the following solution (x, a) :

- $x_i \in \{0, 1\}$ and $a_i \in \{0, 1\}$ for every $i \in \{1, 2, \dots, n\}$, so (C4) is true.
- $x_i = 1$ if and only if $i \in D$.
- if $x_i = 0$, then $a_i = 0$.
- if $x_i = 1$ and i has no neighbours in $V - D$, then $a_i = 1$.
- if $x_i = 1$ and i has at least two neighbours in $V - D$, then $a_i = 0$.

Let us see that $(x, a) \in \Omega$, because since D is dominating, (C1) is satisfied. Next, if $i \notin D$, then $x_i = a_i = 0$ and $\sum_{j \in N(i)} (1 - x_j) \leq |V| - 2$. In this situation it is easy to check that (C2) and (C3) are satisfied. If $x_i = a_i = 1$, then $\sum_{j \in N(i)} (1 - x_j) = 0$ and hence both (C2) and (C3) are true. While if $x_i = 1$ and $a_i = 0$, then $\sum_{j \in N(i)} (1 - x_j) \geq 2$ and again both (C2) and (C3) satisfied.

Note that if $i \in D$ and if i has exactly one neighbour not in D , then $\sum_{j \in N_G(v_i)} (1 - x_j) = 1$ and $x_i = 1$. Therefore (C2) can be rewritten and can be simplified to $|V| \cdot a_i \geq 1$, while (C3) is equivalent to the formula $|V|(a_i - 1) + 1 \leq 0$. Then (C2) implies that $a_i = 1$, but this causes (C3) simplify to $1 \leq 0$, a contradiction. $\square$

Methods

In our recent study, we conducted a comprehensive evaluation of the two algorithms presented in this paper, which we renamed for need of testings and presentation. The algorithm MinimumCertifiedDominating-Set is named `linear_Trees` for short, while the algorithm that uses integer linear programming has name `certified_ILP`, the latter of which is based on linear programming techniques. The algorithms were tested on random trees, and their results and timing were compared using a server computer. The obtained results are given in tables and graphs.

Server computer tests

We performed this assessment on a high-performance server equipped with an Intel(R) Xeon E321xx 3.20 GHz processor, 32.0 GB of RAM, and running the Ubuntu Desktop 22.04.3 LTS system. The algorithms were implemented using SageMath version 9.8 due to its advanced computational features and compatibility with complex algorithmic processes.

The experimental setup generated random trees with varying numbers of vertices, specifically 100, 200, 300, and 400, to fully understand the algorithms' behavior across different scales. Each algorithm was applied to the tree structures, allowing for an analysis of their performance metrics and efficiency levels.

The `certified_ILP` algorithm, which focuses on linear programming, used the `MixedIntegerLinearProgram` package and the GLPK solver as its default computational engine. This integration was crucial for ensuring the accuracy and reliability of the algorithm's linear programming components.

Tables 1 and 2 systematically compile the outcomes of rigorous testing and present a detailed breakdown of performance metrics. The tables offer insights into the efficiency and effectiveness of algorithms for different graph sizes. Results for graphs with 100, 200, 300, and 400 nodes are included, providing a nuanced understanding of the scalability and robustness of the `linear_Trees` and `certified_ILP`.

It can be derived that for the tree on 100 nodes the linear time algorithm was over 40 times faster than the integer linear programming algorithm, while for the 200 nodes – nearly 300 times faster.

The information obtained from these tables and the graph of timings are anticipated to make a substantial contribution to the wider discussion of computational algorithm research, especially in the application of linear and linear programming techniques to intricate tree structures in graph theory.

Enhancing fire safety in urban planning through graph theory

The research on fire protection systems in this study is significant for several reasons. A new model for fire hydrant placement is introduced using graph theory to minimize distances between them. This is crucial because in urban areas, the distance and availability of fire hydrants can significantly affect the effectiveness of fire department response. We use certified dominating sets to enhance fire safety in urban areas by optimizing the placement of hydrants.

	Tests for 100 nodes		Tests for 200 nodes	
	<code>linear_Trees</code> [ms]	<code>certified_ILP</code> [ms]	<code>linear_Trees</code> [ms]	<code>certified_ILP</code> [ms]
Min	0.59	26.10	1.19	337.87
25%	0.76	33.92	1.55	439.22
Mean	1.17	52.19	2.38	675.73
Median	0.88	39.14	1.79	506.80
75%	1.23	54.80	2.50	709.52
Max	1.76	78.29	3.57	1013.60

Table 1. Basic time statistics for trees on 100 nodes and 200 nodes for a server computer.

	Tests for 300 nodes		Tests for 400 nodes	
	linear_Trees [ms]	certified_ILP [ms]	linear_Trees [ms]	certified_ILP [ms]
Min	1.79	3998.46	2.48	977270.79
25%	2.33	5198.00	3.22	1270452.02
Mean	2.38	7996.91	4.96	1954541.57
Median	2.69	5997.68	3.72	1465906.18
75%	3.76	8396.76	5.21	2052268.65
Max	5.37	11995.37	7.44	2931812.36

Table 2. Basic time statistics for trees on 300 nodes and 400 nodes for a server computer.

Furthermore, the distribution system is analysed through the pump stations and wells network to ensure a continuous water supply in the event of a natural fire. This approach not only enhances the ability to extinguish fires, but also reduces certain types of fire damage. Mathematical models are also employed to determine the most cost-effective and efficient location for building supporting infrastructure. This ensures that fire safety devices that are too expensive, redundant, or useless are not purchased by the administrator.

Using mathematical models to determine the optimal placement of hydrant slots and associated infrastructure can eliminate defunct fire safety equipment, resulting in cost savings. This research is essential as it provides a low-cost method to improve our country's fire safety performance.

In addition, it has a significant impact on urban planning and development through the integration of fire safety considerations into the early stages of urban planning, thus promoting a more comprehensive approach to urban planning. The aim is to ensure that new developments have effective fire safety measures from the outset, thus reducing the need for costly upgrades.

Conclusion

This study presents a comprehensive approach for enhancing the efficacy of fire hydrant systems through the application of graph theory. The impact on urban fire safety, economic efficiency, and city planning is significant and provides a plan for more effective and sustainable fire safety strategies in urban environments. This study contributes to the theoretical understanding of graph theory and its practical applications in fire safety in cities.

In this paper we have solved a problem of constructing a linear time algorithm that finds a minimum certified dominating sets in trees. Also, we have identified and corrected an error that appears in the research literature concerning the ILP model for finding a minimum certified dominating set. We have tested and analyzed both algorithms on a server computer.

Data availability

The datasets generated and analysed during the current study are available in the GitHub repository, https://github.com/mmioTk/CertifiedDominatingSets_Algorithms/tree/main/datasets

Code availability

https://github.com/mmioTk/CertifiedDominatingSets_Algorithms/tree/main.

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Author contributions

Conceptualization M.M., visualization J.R. All authors did the investigation, conducted the experiments, analysed the results, prepared the original draft and reviewed the manuscript.

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RESEARCH ARTICLE

Two Approaches to Constructing Certified Dominating Sets in Social Networks

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ABSTRACT Social networks are an important part of our community. In this context, certified dominating sets help to find in networks a group of people, referring as officials, such that 1) for each civilian, there is an official that can serve the civilian, and 2) no official is adjacent to exactly one civilian, to prevent potential abuses. To delve deeper into this topic, this study considers two approaches to the problem of finding certified dominating sets of small cardinality. One approach is to transform an ordinary dominating set, which is subject only to condition 1) given above, into a certified dominating set. The second approach involves constructing such a set from scratch. To compare the two methods, we first studied the computational complexity of the decision problem of whether, for a given graph, the domination number, which is the minimum size of a dominating set, is equal to the certified domination number, that is, the minimum size of a certified dominating set. Next, we constructed and compared the performance of two approximate algorithms that find certified dominating sets, one for each of the two approaches. The effectiveness and efficiency of the proposed algorithms were validated through experiments that compare their results with each other and with a previous study: the linear-time algorithm, which determines the certified domination number for trees and with known results for the domination number in social networks.

INDEX TERMS Algorithmic efficiency, approximation algorithms, certified domination number, computational complexity, domination number, safety, social network theory.

I. INTRODUCTION

Analyzing social networks holds a pivotal position in understanding the modern world across the globe. Currently, billions of people are using Facebook, Instagram, or other tools that help people communicate with their friends, while at the same time, complex networks are constituting and changing while connecting to the mobile phone networks or power networks of smart cities. Meanwhile, researches has been investigating new methods for analyzing these networks, reshaping the landscape of our understanding of such connections. Over the past decade, significant advances have been made in the theory of complex networks, with powerful algorithms being developed to tackle real-world challenges [1], [2], [3], [4].

Recent endeavors in analyzing social networks have led to the development of new methods and tools based on graph

theory, since a social network can be modeled as a graph [5]. Previous studies have addressed, among others, the problem of (k, l) -anonymity of networks as a measure of their privacy against active attacks [6] due to the fact that some of them may contain sensitive information.

The past decades have seen a huge growth in the study of domination-like parameters in graphs, which has become an important issue in exploring the features of networks; namely researches have always been interested in efficient and swift algorithms that determine dominating sets of the least possible cardinality [7], [8], [9], [10], [11], [12]. Recent developments in studying domination parameters have led to the introduction of a certified dominating set [13], in which the authors studied social networks consisting of civils and officials. To ensure that each civil is served by an official and to avoid any abuse from the official towards the civils, two conditions should be met:

- 1) for each civil c there is an official o that can serve c ,

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- 2) whenever any such official o is serving c , there must also be another civil that observes the official.

The smallest cardinality of the set of officials that meets requirements 1) and 2) is the *certified domination number*, denoted $\gamma_{cer}(G)$, where G is the graph model of a network.

Certified domination number has been gaining increasing interest in the literature owing to the practical applications of this parameter. For example, the authors [14] studied placing water pumping stations and wells in a fire safety water network providing that the total cost used to build such utilities is as low as possible and the three conditions are met:

- Each node without a well or pumping station is connected to a node with a pumping station.
- Building a pumping station is profitable in nodes neighboring at least two nodes without a well or a pumping station.
- Each neighbor of a node with a well contains either a well or pumping station.

The algorithms presented in this work are applicable to fire safety water networks.

The first investigations found that the certified domination number is equal to the well-known classical domination number as long as the graph does not contain weak support nodes [13]. The authors of a subsequent paper [15] studied the equality of the two domination parameters; namely, they established conditions for the equality in different classes of graphs and characterized all graphs G for which $\gamma(H) = \gamma_{cer}(H)$ for each induced and connected subgraph $H \neq K_2$ of G . However, there has been little discussion on the computational complexity of the problem of equality of those two dominating parameters for graphs with weak support nodes. This study sheds new light on this problem by formulating its NP-hardness even for graphs with only one weak support node (Theorem 1). Moreover, in the same paper the authors posed the open problem of characterizing trees T for which $\gamma(T) = \gamma_{cer}(T)$. To date this issue has not been mathematically solved. Nevertheless, in [14] the authors presented and tested a linear time algorithm finding a minimum (smallest possible) certified dominating set in trees, which appeared to be much more elaborate compared to the algorithm for the classical domination number in trees.

In addition to the papers mentioned above, the algorithmic aspects of certified dominating sets in graphs have been studied in [16]. Among the many results, this paper provides an Integer Linear Programming (ILP) formulation for the minimum certified domination number. Their main weakness is that they do not attempt to prove the correctness or test their ILP model. Consequently, their result seems to be somewhat inaccurate, since the ILP model does not manage to find the expected result for some graphs, namely a path on four nodes. The authors of [14] have already noted this inconsistency and proposed a new version of an ILP model followed by a correctness proof and tests. In particular, they compared the results returned by the linear-time algorithm for trees and ILP model.

Another weakness of [16] is that the algorithm provides a method for constructing a certified dominating set from any dominating set. Accordingly, this result has neither been proven nor implemented and tested. Furthermore, our findings show that for some input graphs and input dominating sets the algorithm in [16] never stops. This paper takes a new look at this problem, because we explain this issue and provide a correct algorithm, `ClassicToCert`, which constructs a certified dominating set from a minimal dominating set (see Algorithm 1). In this way, this paper formulates a two-step method, which first finds a minimal dominating set (using the well-known greedy algorithm, see [17], [18]) and then applies the `ClassicToCert` algorithm to find a certified dominating set.

Furthermore, this paper presents a new approach to finding minimal certified dominating sets from scratch, the `ApproxCert` Algorithm 3. We conducted this study to compare both methods and to apply the best solution for finding certified dominating sets in social networks. Finally, we compared the results of both methods with the results returned by the linear time algorithm given in [14].

The authors in [19] studied the influence of adding an edge on the certified domination number, whereas [20] is the first study of possibility of changing the value of this domination parameter when a node is removed. Preliminary work has been carried out in recent years regarding various types of certified domination, such as certified perfect domination [21] and connected certified domination [22], [23].

This paper presents and analyzes a new approach for identifying certified dominating sets in social networks. In particular, we discuss the potential possibilities of using well-known heuristics that find minimal dominating sets [17] to construct certified dominating sets. With this in mind we analyze the computational complexity of the decision problem, that is, whether $\gamma(G) \neq \gamma_{cer}(G)$ for a given graph G with exactly one weak support node. This result suggests that finding a polynomial time algorithm that transforms a minimum classical dominating set into a minimum certified dominating set is impossible. Nevertheless, the correlation between these two domination parameters is of interest to many researches, as suggested in previous papers on this topics [13], [15]. Indeed, the `ClassicToCert` algorithm performed slower and returned larger certified dominating sets than the `ApproxCert` algorithm, which is introduced in this paper, and which finds a minimal certified dominating set from scratch. We thoroughly describe and discuss both algorithms in subsequent sections. Our conclusions are presented in the final section of the paper.

II. APPROACH

This section introduces and explains substantial terms being used throughout the paper.

Let G be a simple and undirected graph that represents the model of a social network. We generally follow the notation and terminology of a book on domination in graphs [24].

For a graph G , the set of all nodes is denoted by V and the set of links by E . For a node $v \in V$, the open neighborhood $N(v)$ of v is the set of all nodes adjacent to v , and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v . If $u \in N(v)$, then u is said to be a neighbor of v .

Given graph G , we say that subset $D \subseteq V$ is a dominating set of G if every node belonging to $V - D$ is adjacent to at least one node in D . Equivalently, D is a dominating set of G if $N[v] \cap D \neq \emptyset$ for every $v \in V(G)$. As explained in the Introduction, Condition 1) requires that the set of officials is a dominating set; that is, for every civil c there is an official o that can serve c . The domination number of graph G , denoted by $\gamma(G)$, is the cardinality of a smallest (i.e. minimum) dominating set of G . A dominating set D is minimal if no proper subset of D is a dominating set. In Fig. 1, a set $D_1 = \{x_1, x_3, x_6\}$ is a minimal dominating set, because if we remove any one node from D_1 , it is not dominating. However, D_1 is not minimum; a set $D_2 = \{x_2, x_5\}$ is dominating and has a smaller number of elements than D_1 . Evidently, there is no dominating set of P_6 of cardinality 1; therefore $\gamma(P_6) = 2$.



FIGURE 1. Graph P_6 .

Finding a minimum dominating set in a given graph is algorithmically hard problem [25], while minimal dominating sets can be found by a simple greedy algorithm performing in polynomial time [17], [26].

A subset $D \subseteq V$ is called a certified dominating set of G if D is a dominating set of G and every node belonging to D has either zero or at least two neighbors in $V - D$. These two conditions correspond to requirements 1) and 2) presented in the Introduction. We do not allow an officer o to serve exactly one civil c , since this situation may lead to the abuse of o towards civil c . In the case of any irregularities caused by o with respect to civils, only c may protest. Only one c against o is insufficient to protect c against abuse. Therefore, each o can have either no civils or at least two civils in its neighborhood. The cardinality of a smallest (that is minimum) certified dominating set of G is called the certified domination number of G and is denoted by $\gamma_{cer}(G)$. A certified dominating set of G of minimum cardinality is called a γ_{cer} -set of G . For path P_6 in Fig. 1 $D_1 = \{x_1, x_3, x_6\}$ is not a certified dominating set because both x_1 and x_6 are adjacent to exactly one node not in D_1 . However, $D_2 = \{x_2, x_5\}$ is a minimum certified dominating set, because both elements of D_2 are adjacent to two nodes that do not belong to D_2 .

The degree of node v in G is the number of nodes adjacent to v , that is $d_G(v) = |N(v)|$. A node of degree one is called a leaf of G . If v is a leaf, then its unique neighbor is called the support node of v . If a support node is adjacent to exactly one leaf, then we call it a weak support node; otherwise, we call it a strong support node. For path P_6 in Fig. 1, x_2 and x_5 are weak support nodes. Weak support nodes are the main concern when studying classical domination and certified domination

number, because if a graph does not have a weak support node, then these two dominating parameters are equal. In [13] the authors claimed that

$$\gamma(G) \leq \gamma_{cer}(G) \leq \gamma(G) + sw(G). \quad (1)$$

This implies that if a graph does not contain weak support nodes, then $\gamma(G) = \gamma_{cer}(G)$. Weak support nodes and their influence on the certified domination number are among the major topics of this paper.

III. RESULTS

This section introduces the methodologies and theoretical results of this study. In particular, we focus on the computational complexity of the decision problem of determining whether the domination number is not equal to the certified domination number, and we introduce two approaches for finding certified dominating sets.

A. NP-COMPLETENESS FOR DOMINATION NUMBER VERSUS THE CERTIFIED DOMINATION NUMBER

It is known (see [15]) that for any graph G the certified domination number is never smaller than the domination number because every certified dominating set is also a dominating set. In addition, if the minimum degree of a graph is at least two, the certified domination number and the domination number of the graph are equal [15]. In this section we show that determining whether $\gamma(G) \neq \gamma_{cer}(G)$ is NP-hard, even for graphs with only one leaf and every other node of degree at least two. If a graph contains only one weak support node, then by (1) the certified domination number of this graph is equal to either $\gamma(G)$ or $\gamma(G) + 1$. Formally, we define this decision problem as follows:

PROBLEM: Inequality of $\gamma(G)$ and $\gamma_{cer}(G)$

INSTANCE: Graph G .

QUESTION: Is $\gamma(G) \neq \gamma_{cer}(G)$?

The idea behind our proof is based on a reduction from the 3SAT problem.

PROBLEM: 3SAT

INSTANCE: Given a boolean expression E in conjunctive normal form (CNF), which is the conjunction of clauses, each of which is a disjunction of three distinct literals.

QUESTION: Is E satisfiable?

Theorem 1: It is NP-hard to determine if for a given graph G is $\gamma(G) \neq \gamma_{cer}(G)$ even for graphs with only one weak support node.

Proof: We describe a reduction from 3SAT, which was proven to be NP-complete in [25], to the considered problem. The formula in 3SAT is given in a conjunctive normal form, where each clause contains three literals. We assume that the formula contains the instance of any literal u and its negation u' (otherwise, all clauses containing literal u are satisfied by the true assignment of u).

Given an instance E , the set of literals $U = \{u_1, u_2, \dots, u_n\}$ and the set of clauses $C = \{c_1, c_2, \dots, c_m\}$ of 3SAT, we construct a graph G whose order is polynomially bounded in terms of n and m such that the formula is

satisfiable if and only if $\gamma_{cer}(G) \neq \gamma(G)$. For each variable u_i we construct a copy of $G(u_i)$ of the graph in Fig. 2.

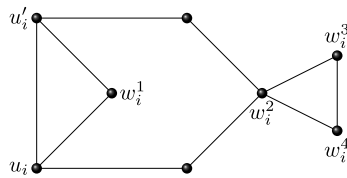


FIGURE 2. The graph $G(u_i)$.

The (m, n) -lollipop graph is obtained by joining a complete graph K_m to a path P_n with an edge. For every clause C_j , we create a copy of $G(C_j)$ of the $(3, 2)$ -lollipop graph with vertices labeled as in Fig. 3. Let X be a copy of K_2 with vertices labeled x_1 and x_2 . All graphs $G(u_i)$ and $G(C_j)$ created so far are disjoint.

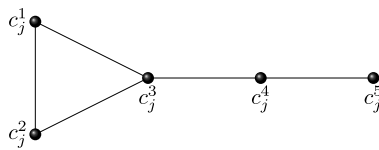


FIGURE 3. The graph $G(C_j)$.

For every clause C_j with literals x, y and z , we create the three edges $c_j^x x$, $c_j^y y$ and $c_j^z z$. If, for example, $C_1 = \neg u_1 \vee u_2 \vee u_3$, then these edges are $c_1^{\neg u_1} u_1$, $c_1^{u_2} u_2$ and $c_1^{u_3} u_3$ as shown in Fig. 4. Moreover add edges $x_2 c_j^x$ for $j = 1, 2, \dots, m$ and $x_2 w_i^i$ for $i = 1, 2, \dots, n$. This completes the construction of G .

Observe that x_1 is the unique leaf in G , x_2 is a weak support node, and $d_G(x_2) = n + m + 1$. Hence, x_2 belongs to every minimum certified dominating set of G (by Observation 3.1 in [13]). If each neighbor of x_2 belongs to a minimum dominating set of G , then $\gamma_{cer}(G) > \gamma(G)$. It remains to be proven that E is satisfiable if and only if each neighbor of x_2 except for x_1 belongs to each minimum dominating set of G .

Graph G contains disjoint induced cycles on three vertices: (c_j^1, c_j^2, c_j^3) , (w_i^1, w_i^3, w_i^4) and (w_i^1, u_i, u_i') . It is easy to verify that at least one node from each cycle is needed for a minimum dominating set of G . Additionally, one node of $N[x_1]$ belongs to each minimum dominating set of G . Therefore, $\gamma(G) \geq 2n + m + 1$.

First, we assume that E is satisfiable and consider a satisfying truth assignment. We take to a dominating set of G each true literal together with w_i^2 to dominate the triangle (w_i^2, w_i^3, w_i^4) for $i = 1, 2, \dots, n$. Since E is satisfiable, each c_j^5 is dominated by a node from $G(u_i)$ and additionally we take to a dominating set of G each node c_j^3 to dominate the triangles (c_1^1, c_2^2, c_3^3) and c_j^4 for $j = 1, 2, \dots, m$. In this

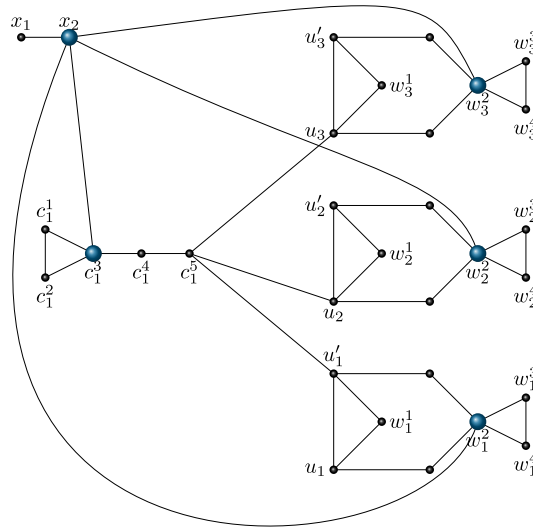


FIGURE 4. Complete construction for the clause $C_1 = \neg u_1 \vee u_2 \vee u_3$.

manner, we construct a dominating set of G of cardinality $2n + m + 1$ and such that each neighbor of x_2 except for x_1 belongs to this dominating set. Therefore without loss of generality, if E is satisfiable, then $\gamma(G) = 2n + m + 1$ and each neighbor of x_2 except for x_1 belongs to each minimum dominating set of G .

Next, we assume that each neighbor of x_2 except for x_1 belongs to each minimum dominating set of G . Then, w_i^2 belongs to every minimum dominating set of G . If both u_i and u'_i belong to a minimum dominating set of G , say D , then $(D - \{w_i^2\}) \cup \{w_i^1\}$ is also a minimum dominating set of G that does not contain w_i^2 , which is a contradiction. Hence, exactly two vertices of $G(u_i)$ belong to each minimum dominating set of G for $i = 1, 2, \dots, n$. Similarly, since c_j^3 belongs to every minimum dominating set of G , then neither c_j^4 nor c_j^5 belongs to any minimum dominating set of G , say D , since otherwise $(D - \{c_j^3\}) \cup \{c_j^2\}$ is a minimum dominating set of G not containing c_j^3 , a contradiction. Hence, only one node of $G(C_j)$, namely c_j^3 belongs to each minimum dominating set of G for $j = 1, 2, \dots, m$. Therefore, $\gamma(G) \leq 2n + m + 1$ and c_j^5 is dominated by a node u_i of u'_i for some i . Now, set true to every literal, which corresponds to a minimum dominating set of G and set false to it, otherwise. Consequently, it follows that the truth assignment defined above satisfies E . This completes this proof. ■

B. TRANSFORMING MINIMAL DOMINATING SETS INTO CERTIFIED DOMINATING SETS

In this section, we study the algorithms that on a basis of a (minimal) dominating set construct a certified dominating set.

The first approach to solve this problem was presented in [16], Algorithm Construction of CFDS. The authors

introduced an algorithm that transforms a dominating set into a certified dominating set. Let us briefly describe the Algorithm Construction of CFDS [16]. Throughout the algorithm, D_1 is a dominating set that is transformed into a certified dominating set. $U \subseteq D_1$ is the set of elements of D_1 which are adjacent to exactly one node outside D_1 . If $U = \emptyset$, the algorithm stops because D_1 is a certified dominating set. Therefore, the aim of the Construction of CFDS algorithm is to eliminate nodes from U . This is performed in the following manner. Let v be the unique neighbor of u belonging to $V(G) - D_1$.

- if v is a leaf, then add v to D_1 ;
- otherwise add v to D_1 and remove u from D_1 .

Consequently, $u \notin D_1$. However, the authors [16] overlooked some problems that arose after the implementation of the algorithm. In fact, there are input graphs and dominating sets for which it never stops. In this section two examples are presented.

- Let G be a cycle on three nodes, u_1, u_2, u_3 and let $D_1 = \{u_1, u_2\}$ be the input dominating set. Then $U = \{u_1, u_2\}$. After applying the main loop of the Construction of CFDS for u_1 , the situation is roughly the same, because $D_1 = \{u_3, u_2\}$ and $U = \{u_3, u_2\}$. Consequently, this algorithm does not end.
- Even if the first dominating set, namely D_1 , is minimum, the algorithm still does not guarantee that it will stop. Consider a path $P_7 = (u_1, u_2, \dots, u_7)$ and let $D_1 = \{u_2, u_3, u_6\}$. Then $U = \{u_2, u_3\}$. After applying the rule from the algorithm to u_2 , we obtain $D_1 = \{u_1, u_2, u_3, u_6\}$ and $U = \{u_3\}$. We then add u_4 to D_1 and remove u_3 , so $D_1 = \{u_1, u_2, u_4, u_6\}$ and $U = \{u_2\}$. Next the algorithm removes u_2 from D_1 and adds u_3 , whereas in the next loop, it can do the opposite: remove u_3 from D_1 and adds u_2 . U is never empty, and the algorithm never stops.

It appears that two simple conditions, as presented in the algorithm in [16], are insufficient to ensure that this algorithm works properly and solves the problem. In what follows, we present Algorithm 1, *ClassicToCert*, which constructs a certified dominating set based on a minimal dominating set. Some algorithms that work in a reasonable time finding a minimal dominating sets of a graph can be found, for example, in [17]. Our algorithm always stops, as we prove in Theorem 2.

Note: If one of the components of G is K_2 , then we assume that one node of K_2 is a leaf, while the second is a support node. We do not let a node be both a leaf and a support node.

Theorem 2: Algorithm 1 *ClassicToCert* returns a certified dominating set of G .

Proof: To prove that the algorithm finds the certified dominating set it suffices to prove that at the end of performing, the algorithm set D is a dominating set (note that D may not be a minimal dominating set during the performance of the algorithm and at the end) and $U = \{u \in D : |N(u) \cap (V - D)| = 1\}$ is an empty set.

Algorithm 1 *ClassicToCert* Function

Input: A simple graph G

Output: Certified dominating set D

```

1:  $D \leftarrow$  a minimal dominating set of  $G$ 
2:  $U \leftarrow \{u \in D : |N(u) \cap (V - D)| = 1\}$ 
3: while  $U \neq \emptyset$  do
4:   Let  $u \in U$  and let  $v$  be the unique node in  $N(u) - D$ 
5:   if  $u$  is a leaf then
6:      $D \leftarrow (D \cup \{v\}) - \{u\}$ 
7:   else if  $u$  is a support node then
8:     if  $v$  is a leaf then
9:        $D \leftarrow (D \cup \{v\})$ 
10:    else
11:      remove from  $D$  all leaves incident with  $u$ 
12:    end if
13:  else
14:    if  $N[v] \cap D = \{u\}$  then
15:       $D \leftarrow (D \cup \{v\}) - \{u\}$ 
16:    else
17:       $D \leftarrow D - \{u\}$ 
18:    end if
19:  end if
20:   $U \leftarrow \{u \in D : |N(u) \cap (V - D)| = 1\}$ 
21: end while
22: return  $D$ 

```

D is a dominating set in line 1 of the algorithm. During the performance of the algorithm, D is modified. In particular, in line 9 a node is added to D , so D is still a dominating set.

If line 6 is performed, D is still a dominating set because u , which is removed from D , is a leaf and v , which is a support node adjacent to u , is added to D .

If the condition in line 10 is true, then u is a support node. Since $u \in D$, we remove the leaves incident with u from D and D remains a dominating set.

If condition in line 13 is true, then $\deg(u) > 1$, so $|N(u) \cap D| \geq 1$. Moreover, if v is dominated by another node from D (except for u), then $D - \{u\}$ is also a dominating set. If u is the unique neighbor if v in D , then we remove u from D and add v to D to ensure D remains a dominating set. Consequently, D is a dominating set when performing the algorithm.

Now, we prove that reaching $|U| = 0$ is inevitable when using this algorithm. If U calculated in line 2 is empty, then D is a certified dominating set and the algorithm quits.

In what follows, we prove that if U is not empty in line 3, then the size of U can increase by a finite number of times. The algorithm in [16] does not guarantee this because U may have the same number of elements through several **while** loops and hence U never reaches the condition $|U| = 0$. For the two examples studied above, from the perspective of performing the algorithm, in each **while** loop a processed node quits U , however a new node enters U . In this way, the algorithm presented in [16] never stops.

TABLE 1. Influence of modifications of D on the number of nodes in U and in R .

line no.	influence on U	influence on R
line 6	may increase (situations 2, 3) or may decrease	may increase (neighbors of v may transfer from U to R)
line 9	decreases by 1	increases by 2 (u and v)
line 11	decreases by 1	does not change
line 15	may increase (situation 1)	may decrease (neighbors of u may transfer from R to U)
line 17	or may decrease	

We denote by $R = \{u \in D : N[u] \subseteq D\}$, that is the set of all nodes of D whose each neighbor is also in D . Note that since D is minimal (line 1), $R = \emptyset$ at the beginning of the algorithm. In addition, if $u \in R$, then $u \notin U$, but if one of its neighbors is removed from D , then u is added to U . In general, a new node u is added to U as a result of the following three situations:

- **Situation 1:** If $u \in R$ and one of its neighbors is removed from D , then $u \in U$ in the next loop.
- **Situation 2:** If $|N[u] - D| = 2$ and exactly one of the neighbors of u is added to D , then $u \in U$ in the next loop.
- **Situation 3:** If u , previously not in D and with $|N(u) - D| = 1$, is added to D , then $u \in U$ in the next loop.

Table 1 summarizes the influence of the modifications of D on the number of nodes in U and in R .

We then proceed with the proof by considering several claims.

Claim 1: If a support node belongs to D , then is never removed from D .

If u is a support node, then only the condition in line 7 is true. In lines 8–12 u is never removed from D .

Claim 2: If support node u does not belong to D , then u is added to D before the end of the algorithm.

If a support node u does not belong to D , then since D is a dominating set, all leaves adjacent to u belong to D and also to U . Hence, for each such leaf line 6 is performed, and therefore eventually u becomes an element of D .

Claim 3: U may increase in situations 2 and 3 a finite number of times.

Since by Claims 1 and 2 each support node is added to D and never removed from D , the statement in line 5 is true a finite number of times; therefore, the result follows.

Claim 4: If $u \in R$, then u is a support node or a neighbor of a support node.

By column 3 of Table 1, the number of nodes in R increases only when lines 6 and 9 are performed, thus, the result follows.

Claim 5: Node u can be added to R at most once.

We assume $u \in R$ is a support node. Then, u is removed from R if a neighbor of u , say w , belongs to U and either line 15 or 17 is executed. In this way $u \in U$ and either another neighbor of u is removed from R or line 11 is performed. Since u is a support node and by Claim 1, u is added to D and never removed from D , w can not be added to D again (the condition

in line 14 is never true for $v = w$). Therefore if u is a support node, the result follows.

We assume now that $u \in R$ is a leaf. Then, u is removed from R if the condition in line 10 is true. By reasoning similar to the above, the conditions in lines 7–8 will never be true, and thus the result follows.

If $u \in R$ is neither a leaf nor a support node, then by Claim 4 u is a neighbor of a support node. Hence, u is adjacent to a node in D until the completion of the algorithm. Therefore, if at some point $u \notin D$, then u will never be an element of D again. In addition, by reasoning similar to that above, if a neighbor of u does not belong to D , then it cannot be added to D again.

Claim 6: U may increase by situation 1 a finite number of times.

This result is a consequence of Claim 5.

Since all three situations can occur a finite number of times (see Claims 3 and 6), new nodes can be added to U finite number of times. Therefore, the algorithm always stops. At the end of the algorithm D is a dominating set, and since $U = \emptyset$, D is a certified dominating set. ■

C. CONSTRUCTING A MINIMAL CERTIFIED DOMINATING SET FROM A SCRATCH

To investigate the problem of transforming dominating sets into certified dominating sets in more detail in this section we introduce the `ApproxCert` function, which determines a minimal certified dominating set from scratch. This allows us to compare the efficiency and performance of the two approaches.

In the `ApproxCert` algorithm, every node stores the local variables listed below. Their values depend mainly on which node v belongs to the emerging minimum certified domination set ($v.D = 1$) and which node does not ($v.D = 0$). At the beginning of the algorithm D in an empty set. The algorithm terminates when the nodes belonging to D form a minimal certified dominating set (see Theorem 3).

For node v the local variables are as follows:

- $v.D = 1$ if v belongs to $D \subseteq V(G)$ and $v.D = 0$ otherwise.
- $v.dom$ is the number of nodes in the closed neighborhood of v belonging to D .
- $v.out$ is the number of neighbors (open neighborhood) of v not in D .
- $v.pow$ is the number of nodes in the closed neighborhood of v with $dom = 0$ minus the number of (neighbors of v in D and with $out = 2$), and minus one if $v.out = 1$.
- $v.sta = L$ if v is a leaf, $v.sta = S$ if v is a support node and $v.sta = 0$ otherwise.

We assume that all the variables are set to 0 before the launching of the algorithm, so in particular $D = \emptyset$.

Since the support nodes always belong to every certified dominating set, the function `Initialize` (Algorithm 2) sets $v.D = 1$ and $v.sta = S$ for each support node v in G . In the first step, it identifies leaves, that is nodes of degree 1

Algorithm 2 Initialize Function**Input:** A simple graph G **Output:** G with support nodes added to D

```

1:  $v.sta \leftarrow L$  if and only if  $v$  is a leaf
2:  $v.sta \leftarrow S$  if and only if  $v$  is a support node
3:  $v.D \leftarrow 1$  if and only if  $v$  is a support node
4: for  $v \in V(G)$  do
5:    $v.out \leftarrow |\{y \in N(v) : y.D = 0\}|$ 
6:    $v.dom \leftarrow |\{y \in N[v] : y.D = 1\}|$ 
7: end for
8: for  $v \in V(G)$  do
9:    $v.pow \leftarrow |\{y \in N[v] : y.dom = 0\}|$ 
10:  for  $y \in N[v]$  do
11:    if  $y.D = 1$  and  $y.out = 2$  then
12:       $v.pow \leftarrow v.pow - 1$ 
13:    end if
14:  end for
15:  if  $v.out = 1$  then
16:     $v.pow \leftarrow v.pow - 1$ 
17:  end if
18: end for
19: return  $G$ 

```

(see line 1) and assigns them status L . Next, if v is a neighbor of a leaf, then v is a support node, so $v.sta = S$ and also $v.D = 1$ meaning v is added to D (lines 2–3).

The next lines of Algorithm 2 for each node v count the number of its neighbors not in D (value stored in variable out) and in D (values stored in dom), see lines 4–7, and then, on the basis of values of variables out and dom it determines the values of the variable pow . In general, $v.pow$ is equal to the number of non-dominated nodes in the closed neighborhood of node v (line 9). However, if v has a neighbor in D that has two neighbors outside D , then $v.pow$ decreases by one, because no node in D can have exactly one neighbor not in D (lines 11–13). In this case, adding v to D can cause problems when transforming D into a certified set. Similarly, if v has exactly one neighbor not in D , then also $v.pow$ decreases by one (line 16), by analogous reasoning.

Function ApproxCert (Algorithm 3) is the primary function. First, it uses the Initialize function (Algorithm 2) to identify and include in D all the support nodes and set the values of the local variables of each node. Next, in the main loop (lines 2–7), it chooses node x with the greatest value of variable pow and adds it to D as long as there are some nodes that are not dominated (have $pow = 0$). In lines 5–6 the algorithm updates the values of dom and out in the neighborhood of x as well as the values of $x.pow$ and all neighbors of x at a maximum distance of 2 from x .

In the second part of ApproxCert (Algorithm 3), in lines 8–13 the algorithm checks whether D is a minimal dominating set. If $v \in D$ and at least one of its neighbors is in D ($v.dom > 1$) and every neighbor is in D or is dominated by

Algorithm 3 ApproxCert Function**Input:** A simple graph G **Output:** Minimal certified dominating set D

```

1:  $G \leftarrow \text{Initialize}(G)$ 
2: while  $\exists v \in V v.D = 0$  do
3:    $x \leftarrow$  a node with the maximum value of  $pow$ 
4:    $x.D \leftarrow 1$ 
5:    $\forall y \in N[x]$  update  $y.dom$  and  $y.out$ 
6:    $\forall y \in N_2[x]$  update  $y.pow$ 
7: end while
8: for  $v \in V(G)$  do
9:   if  $v.D = 1$  and  $v.dom > 1$  and  $\forall y \in N(v)(y.D = 1$ 
    or  $y.dom > 1)$  then
10:      $v.D \leftarrow 0$ 
11:      $\forall y \in N[v]$  update  $y.dom$  and  $y.out$ 
12:   end if
13: end for
14: for  $v \in V(G)$  do
15:   if  $v.sta \neq S$  and  $v.D = 1$  and  $v.out = 1$  then
16:     for  $y \in N(v)$  do
17:       if  $y.D = 0$  then
18:          $y.D \leftarrow 1$ 
19:          $v.D \leftarrow 0$ 
20:          $\forall z \in N[y] \cup N[v]$  update  $z.dom$  and  $z.out$ 
21:       end if
22:     end for
23:   end if
24: end for
25: for  $v \in V(G)$  do
26:   if  $v.sta = S$  and  $v.out = 1$  then
27:     for  $y \in N(v)$  do
28:       if  $y.D = 0$  then
29:          $y.D \leftarrow 1$ 
30:       end if
31:     end for
32:   end if
33: end for
34:  $D = \{v \in V : v.D = 1\}$ 
35: return  $D$ 

```

at least two nodes in D (line 9), then we remove v from D and update the values of the local variables of v and its neighbors (lines 10–11).

The third part of Algorithm 3 concerns nodes in D that have exactly one neighbor not in D (lines 14–24). Namely, if $v \in D$ and v is not a support node and has exactly one neighbor not in D , say y , then we add y to D , remove v from D and update the values of the local variables (lines 15–23). Since v is not a support node, y is not a leaf; therefore, after removing v from D and adding y to D , v is dominated by y and y has at least two neighbors not in D . The last statement in this loop (line 20) updates the values of the local variables of v , y and their neighbors.

If v is a support node, then $v \in D$. Additionally, if v has exactly one neighbor not in D , say y , then v is a weak support node. In this case we add y to D (lines 25–33).

Theorem 3: *The ApproxCert function (Algorithm 3) finds a minimal certified dominating set in a polynomial time.*

Proof: It is easy to see that the Initialize function (Algorithm 2) operates in time $O(n^2)$, while ApproxCert function (Algorithm 3) operates in at most $O(n^3)$, where n is the size of the input graph.

The **while** loop of the ApproxCert function (lines 2–7) ensures that when it finishes, every node v has $v.dom > 0$, which means that every node of G is a neighbor of a node belonging to D . Therefore, D is a dominating set in G . It is worth noting that no leaf is added to D , since if u is a leaf, then $u.pow \leq 0$ and no node with non-positive pow can be added to D .

Lines 8–13 verify whether D is a minimal dominating set. If not, v is removed from D . Hence, after performing lines 1–13 D is a minimal dominating set of G .

After performing lines 1–13 of the ApproxCert function, every support node v has $v.D = 1$. Additionally, if no connected component of G is isomorphic to K_2 , then no leaf of G belongs to D . Note that for K_2 , the Initialize function adds to D both nodes of K_2 and they are in D until the end of the algorithm. Therefore, in what follows we assume that no connected component of G is isomorphic to K_2 .

Lines 14–33 deal with the demand that no node in D can have exactly one neighbor not in D . Let v be a node for which the condition in line 15 is true. Then v has exactly one neighbor not in D , so D is not certified domination. Moreover, v is neither a leaf nor a support node. In addition, y in line 15 is neither a leaf nor a support node. Hence, $deg(v) > 1$ and $deg(y) > 1$. Therefore, since $v.out = 1$, every other neighbor of v belongs to D . Further, since D is a minimal dominating set, v is the unique neighbor of y that belongs to D (otherwise line 9 would be true for v , and hence v would be removed from D). In this case we add y to D and remove v from D . In this way v is not in D , y belongs to D and has at least two neighbors outside D , and every other neighbor of v has one more neighbor not in D . Note that after swapping v and y , no neighbor of y has exactly one neighbor outside of D , since for such a neighbor z we would have $N[z] \in D$, so D would not be a minimal dominating set after performing lines 1–13. Hence, whenever v is a node for which the condition in line 15 is true, the total number of nodes in D with exactly one neighbor not in D decreases after performing lines 16–20. Moreover, since no node from $N(y)$ belongs to D , D is still a minimal dominating set of G .

Hence, after performing lines 1–24 D is a minimal dominating set for which no node in D , except for possibly support nodes, has exactly one neighbor not in D .

Lines 25–33 solve the problem of support nodes with exactly one neighbor not in D . Let v be a node for which the condition in line 26 is true. Then, v is a weak support node, and the unique neighbor of v that is not in D is a leaf. In this case we simply add the leaf to D . Again, the total number of

nodes in D with exactly one neighbor not in D decreases after performing lines 25–33 and eventually it is equal to 0. Hence D is a minimal certified dominating set of G . ■

IV. DISCUSSION

In this section we present the results obtained during a comprehensive evaluation of the two algorithms presented in this paper: ClassicToCert and ApproxCert.

First, the algorithms were tested on random trees, and their results and timing were compared with the linear time algorithm presented in [14]. Since the timings and results of ApproxCert appeared to be much better than those of ClassicToCert, we applied this algorithm to social networks Google+ and Pokec, which are publicly available at <http://davidchalupa.github.io/research/data/social.html>.

We performed these assessments on a computer equipped with an Intel(R) Xeon E321xx 3.20 GHz processor, 32.0 GB of RAM, and running the Ubuntu Desktop 22.04.3 LTS system. The algorithms were implemented using the language R and a package igraph for operations on graphs. We chose this software because it provides one of the most rapid methods to implement and test graph algorithms. Although the performance of the algorithms was slower than that implemented in C++, we decided to use R because our aim was to compare the two strategies for finding small certified dominating sets.

The input dominating set for the ClassicToCert algorithm was determined using a fast greedy algorithm for dominating sets, as described in [17] and [18].

A. TESTS FOR TREES

First, we performed test on trees using the linear time algorithm for trees [14] as a reference for the results and timings of the ClassicToCert and ApproxCert algorithms because the linear time algorithm for trees always returns the best possible solution. Therefore, the size of a certified dominating set returned by this algorithm is always equal to the certified domination number of the tree. For testing purposes, we generated random trees with the number of nodes varying from 1000 to 3000 in steps of 100. Each algorithm was run once for a given input tree and the input tree was the same for each algorithm.

Unexpectedly, the sizes of the certified dominating sets returned by both ApproxCert and the linear time algorithm for the trees were equal. Consequently, for every random tree generated in this test, the ApproxCert algorithm managed to find the best possible solution, which was the smallest certified dominating set. Surprised by this result, we generated and tested hundreds of trees, whereas we managed to find two trees for which the ApproxCert algorithm returned a worse solution than the exact algorithm.

In Fig. 5, we present the sizes of certified dominating sets returned by ClassicToCert and ApproxCert algorithms.

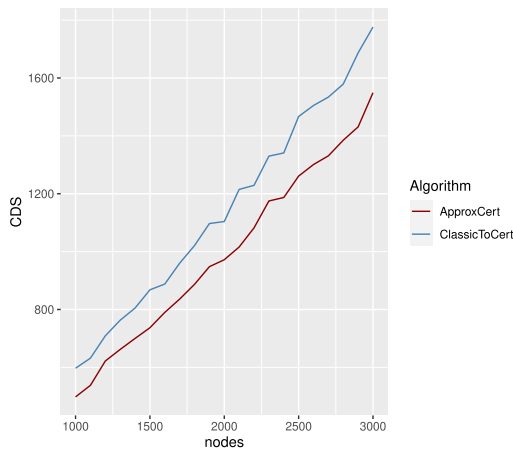


FIGURE 5. Size of certified dominating sets (CDS) for trees.

While the ApproxCert algorithm returned the optimal value in almost all cases, the results obtained by the ClassicToCert algorithm were approximately from 12% to 20% worse than optimal (mean and mode 15%). Perhaps these results of ClassicToCert could be better providing as an input to this algorithm a minimal dominating set containing necessarily all support nodes. However, the approximation ratio of this a major disadvantage regarding the ClassicToCert algorithm was its operating time, see Fig. 6.

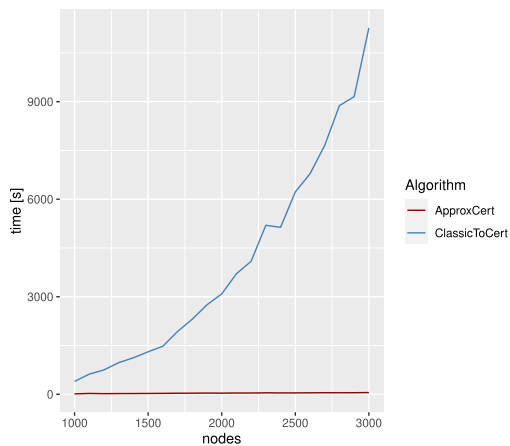


FIGURE 6. Operating times for trees.

The computational complexity of the ClassicToCert algorithm has not been clearly stated in the proof of Theorem 2 because the number of loop cycles of the **while** loop depends on an input graph G and its minimal

dominating set D . The most time-consuming operation is the update of U in line 20 of Algorithm 1. Inevitably, the computational complexity is $O(n^2)$, where n is the number of nodes. Although there are some possibilities to reduce this complexity to $O(\Delta^2)$ by updating U only for nodes in the neighborhood of nodes that were added or removed from D , we decided not to make any unnecessary amendments to Algorithm 1. Our original intention was to implement the Construction of CFDS algorithm presented in [16]. Notwithstanding the long processing time of the ClassicToCert algorithm, we believe that our findings compare well with those of the ApproxCert algorithm and the NP result presented in Theorem 1.

In addition, the fact that the timing of ClassicToCert depends not only on the size of an input graph but also on its structure can be clearly seen in Fig. 6 because the timing graph is not smooth.

In Fig. 7, we compare the timings of the ApproxCert algorithm and the linear time algorithm presented in [14].

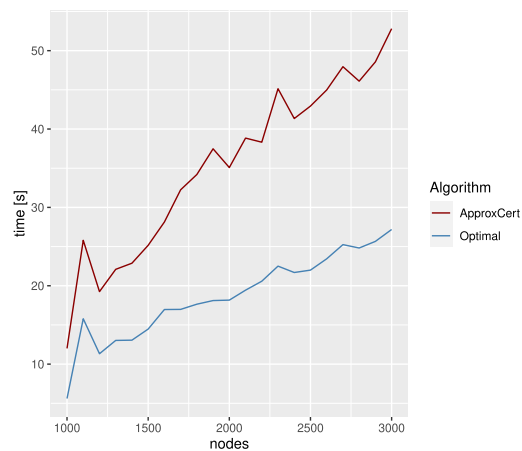


FIGURE 7. Operating times for trees.

Each of the three algorithms was run on the same set of trees; however, for better readability we present their timings on two separate charts (Fig. 6 and Fig. 7). The results suggest that although the theoretical computational complexity of the ApproxCert algorithm is $O(n \cdot \Delta^3)$, it runs in a nearly linear time with respect to the size of a tree. However, the algorithm is susceptible to overall structure of a tree because the graph line is not smooth.

B. TEST FOR SOCIAL NETWORKS

In our last experiment, we applied the ApproxCert algorithm to the Google+ and Pokec social network graphs.

Table 2 summarizes the results of the ApproxCert algorithm. The algorithm was run once for each social network. The longest time, over an hour, took the test gplus_10000, whereas the shortest (2.5 s) took the test

TABLE 2. Experimental results of the ApproxCert algorithm for samples of social networks Google+ and Pokec. All of these network samples are publicly available at: <http://davidchalupa.github.io/research/data/social.html>.

G	$n(G)$	$\gamma(G)$	ApproxCert(G)	$sw(G)$
gplus_200	200	19	19	3
gplus_500	500	42	43	6
gplus_2000	2000	170	171	36
gplus_10000	10000	861	861	209
pokec_500	500	16	16	1
pokec_2000	2000	75	75	6
pokec_10000	10000	413	415	83

gplus_200. To the best of our knowledge no exact values of the certified domination number have been known to date for the Google+ and Pokec networks. For this reason we compared the results returned by the ApproxCert algorithm with known values of the domination number (column $\gamma(G)$). The last column, $sw(G)$ gives the number of weak support nodes of the networks. Considering the bounds on the certified domination number (1), we obtain a general estimate of the certified domination number.

Our test revealed that the results returned by the ApproxCert algorithm were very close to the left bound of the certified domination number. Interestingly, we can be sure that the algorithm managed to find the certified domination number for four graphs because the result given by ApproxCert is equal to $\gamma(G)$ in four cases. In particular, the algorithm managed to find the smallest possible certified domination number for network gplus_10000, whereas the bounds of inequality chain (1) suggest that this number can be as large as 1070. In the case of the other three graphs, ApproxCert(G) is greater than $\gamma(G)$ by one or two. However, because we do not know the exact value of the certified domination number of these graphs, the results should be interpreted with caution, as these might also be exact values.

Generally, the ApproxCert algorithm appeared to return significant results when applied to social networks in reasonable time. We believe that this algorithm can be implemented in other programming languages and applied to a greater number of social networks.

V. CONCLUSION

This paper highlights two possibilities for finding certified dominating sets of small cardinality. First, we have demonstrated that stating whether for a given graph the domination number is not equal to the certified domination number is an NP-hard question.

Next, considerable progress has been made in finding certified dominating sets of small cardinalities. We have explored two potential approaches: starting from a minimal dominating set and from scratch.

The first algorithm, ClassicToCert 1, transforms a minimal dominating set into a certified dominating set. While testing this algorithm, a number of potential weaknesses have been found, namely, time complexity and results that are worse than optimal. Despite this, we believe our work could be the basis for future investigations on this topic.

The second algorithm, ApproxCert 3, constructs a minimal certified dominating set from scratch in a polynomial time. Our tests have proven that this algorithm has a potential to be a powerful tool for identifying certified dominating sets in social networks. We have obtained satisfactory results, thereby proving that the algorithm is both effective and efficient when applied to complex problems, comparing to Algorithm ClassicToCert.

The present findings have important implications for solving versatile social networks problems. Our data suggest that certified dominating sets of small cardinality should be constructed from scratch, in contrast to transforming dominating sets into certified dominating sets. We hope that our research will be valuable for identifying certified dominating sets in social networks.

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