Summary of professional accomplishments

Thomas Zlosnik

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Contents

1 Name and surname

Thomas George Zlosnik

2 Diplomas, scientific degrees

- **DPhil (PhD) in Astrophysics** – October 2005 – September 2008
  Institution: University of Oxford, United Kingdom Department of Astrophysics
  PhD thesis: Cosmological consequences of modified theories of gravity
  Supervisor: Professor Pedro Ferreira
- **MPhys** – October 2001 – September 2005
  Institution: University of Oxford, United Kingdom Department of Physics

3 Information on previous employment in scientific institutions

- **Postdoctoral Fellow** – October 2008 – November 2011
  Institution: Perimeter Institute for Theoretical Physics, Waterloo, Canada
- **Postdoctoral Fellow** – May 2012 – May 2014
  Institution: Theoretical Physics Department, Imperial College London, United Kingdom
- **Vědecký Pracovník (Scientist)** – September 2016 – September 2022
  Institution: Czech Academy of Sciences, Prague, Czechia.
- **NCN Polonez Bis Fellow** – October 2022 – Present Day
  Institution: Institute of Theoretical Physics and Astrophysics, University of Gdańsk, Poland

4 Description of the achievement according to Art.219 Paragraph 1 Point 2 of the Act

4.1 Title of the achievement

A single-themed series of publications entitled *Lorentz symmetry in gravitation and the dark matter problem.*

4.2 List of selected publications

Publications that belong to the series will be cited with the letter [H], publications involving the applicant that do not belong to the series will be cited with the letter [O], and citations to external works not involving the applicant will be cited with the letter [E]. The list of thematically related publications:

1. *An introduction to the physics of Cartan gravity* [H1]
   Hans Westman, Tom Zlosnik
   Annals Phys. **361** 330-376 (2015)
2. *Spacetime and dark matter from spontaneous breaking of Lorentz symmetry* [H2]
   Tom Zlosnik, Federico Urban, Luca Marzola, and Tomi Koivisto
   Class. Quant. Grav., **35**(23), 235003 (2018)
3. *Gravitational alternatives to dark matter with tensor mode speed equaling the speed of light* [H3]
   Constantinos Skordis and Tom Zlosnik
   Phys. Rev. D, **100**(10), 104013 (2019)
4. *New Relativistic Theory for Modified Newtonian Dynamics* [H4]
   Constantinos Skordis and Tom Zlosnik
   Phys. Rev. Lett., **127**(16), 161302 (2021)

5. *Aether scalar tensor theory: Linear stability on Minkowski space* [H5]
   Constantinos Skordis and Tom Zlosnik
   Phys. Rev. D, **106**(1), 104041 (2022)

6. *Paths to gravitation via the gauging of parameterized field theories* [H6]
   Tomi Koivisto and Tom Zlosnik
   Phys. Rev. D, **107**(12), 124013 (2023)

7. *Hamiltonian formulation of gravity as a spontaneously-broken gauge theory of the Lorentz group* [H8]
   Mehraveh Nikjoo and Tom Zlosnik
   Class. Quant. Grav., **40**(1), 015007 (2023)

8. *Aether scalar tensor theory: Hamiltonian Formalism* [H7]
   Marianthi Bataki, Constantinos Skordis, and Tom Zlosnik
   Phys. Rev. D, **110**, 044015 (2024)

9. *Dynamical system analysis of cosmological evolution in the Aether scalar tensor theory* [H9]
   João Luís Rosa and Tom Zlosnik
   Phys. Rev. D, **109**(2), 024018 (2024)

5 Presentation of significant scientific activity

5.1 Introduction

We are currently in an era of enormous advances in cosmology, where the distribution of large-scale structure in the universe and the relic-radiation from the big bang are being measured with unprecedented precision. This data contains a wealth of information about the evolution and composition of the universe, from the earliest known times to the present day.

Meanwhile, the previous century has seen the parallel rise of two physical theories which each possess incredible explanatory power. The first of which is Einstein's General Relativity (GR) which describes gravitation in terms of the curvature of spacetime (indeed identifying the gravitational field with spacetime itself). This theory has passed every known test in the solar system and has recently received further support from the direct detection of gravitational waves as predicted by the theory. The second is the Standard Model (SM) of particle physics, which not only had extraordinary success in accounting for particle data at the time of its creation but has successfully predicted the existence of a variety of particles, the most recent being the Higgs boson.

However, it has been found on scales ranging from galaxies to clusters of galaxies to the very largest scales in the cosmos that the application of GR along with the SM yields predictions at odds with the data. In each of these cases, visible matter behaves as if there were some additional, unseen matter present, sourcing the gravitational field. Remarkably, the cosmic abundance of this matter – termed dark matter (DM) – is several times that of the known matter of the SM.

The DM problem has spurred a great deal of theoretical activity. Cosmological and astrophysical constraints suggest that it is not comprised of ingredients from the SM. The majority of the search for DM candidates has been in the form of new particle physics beyond the SM– for example, sterile neutrinos, the lightest supersymmetric partner, and the axion. A significant amount of effort has gone in into detecting signatures of DM particle candidates. To this day, there does not exist widely accepted evidence for such signatures.

All known effects due to DM are via its effect on the gravitational field and hence via the motion of matter, for example the orbits of stars in galaxies or the gravitational lensing of light around galaxies. As such, it remains a possibility that the DM effect is something that arises not from new particles, but an alternative model of the gravitational field produced by known matter sources. In a sense this is not a new situation in physics. In 1846, the planet Neptune was discovered, with its existence having been postulated earlier by using Newton's theory of gravity to model its perturbative effect on the orbit of the planet Uranus. Decades later, the planet Vulcan was hypothesized as the source of the disagreement between the observed perihelion precession of Mercury and the predictions of Newton's theory. However, famously the resolution to this problem came not from new matter but from the modification to Newton's law implied by GR.

There is another, independent open problem in cosmology that echoes the gradual discovery of the contents of the solar system. Given the known abundances of SM particles in the universe and DM as well as 'standard' couplings between matter and gravity, the theory of GR predicts the geometry of the universe in the past, with the universe becoming of higher and higher temperature and spacetime curvature at earlier times until GR's classical description of spacetime likely breaks down. This forms an 'expansion' history of the universe, spanning about 13.7 billion years in time [E110]. In the night sky we detect a relic of radiation from the early universe called the cosmic microwave background (CMB). Its temperature is - up to small deviations - uniform throughout the entire sky. This is a problem for the cosmic history described above as the CMB photons created in the universe that eventually reach earth from opposite directions in the sky would have emerged from regions in space that were never in causal contact with each other according to GR and the assumed matter content. This is the CMB horizon problem A solution to this problem takes the form of cosmic inflation - a period of extremely rapid expansion in the early universe created by new degrees of freedom in physics. Not only does this solve the CMB horizon problem but models of inflation additionally appear to provide - via the quantum vacuum fluctuations of the new degrees of freedom -a good explanation for the origin of the primordial inhomogeneities in the universe's matter distribution that later evolve to become galaxies. Current data doesn't discriminate between cosmic inflation arising from a new matter field in physics [E80], the effect of non-minimal coupling between known matter and gravity [E69], and modifications to Einstein's theory [E12].

The Habilitiation series is concerned with two sets of models co-developed by the applicant which seek to extend the theory of General Relativity to incorporate the dark matter effect as a manifestation of gravitation. All the models share the feature of spontaneously breaking Lorentz symmetry at the level of their field equations. This has the effect of defining a preferred time direction/state of rest at each point in spacetime. The models are divided into two types:

- **Type A:** These are models where the gravitational degrees of freedom arise from a Cartan-geometric description of spacetime rather than a description based on Riemannian geometry. Nonetheless, remarkably, the spacetime metric and extensions to GR including an effective DM-like matter component will be shown to emerge.
- **Type B:** These are models where the geometric description of gravity remains much as in GR but with the addition of new fields in the gravitational sector. Alongside the metric tensor $g_{\mu\nu}$ that is present in GR, a pair of fields (ϕ, A_μ) are additionally posited. The effect of these new degrees of freedom will be shown to resemble DM in cosmology but, remarkably, more closely resemble a breakdown of the Newtonian limit of GR in the regime of weak gravitational fields.

5.2 Motivation and goals

The goal of the series of publications is the exploration of the idea that effects attributed to DM may, rather, be due in some part to an interaction between known matter and the gravitational field/spacetime that is not accounted by GR. This will involve the exploration of two models of gravitation that were co-created by the applicant. Each model can be considered to be an extension of GR in the sense that each model can resemble GR in certain regimes so that it passes the same experimental constraints that GR does.

In the standard model of cosmology, DM on large scales currently outweighs (by mass) the entirety of the matter of the SM by a factor of around six [E110], and so an advance in our understanding of the effects attributed to DM would mark significant progress in our understanding of the constituents of the universe. In looking to see whether some of this effect may be due to - in effect - new degrees of freedom in the gravitational sector, any compelling results in this direction would necessarily increase our understanding of the gravitational field/spacetime. This could be of vital importance for open questions such as the nature of gravitation as a quantum theory. Though an explanation for DM in terms of extensions to GR is the 'road less travelled', the importance of the DM effect in our universe alongside the inevitable theoretical advances that would accompany success in this research area constitute a strong motivation.

The series of publications presented here regard two independent extensions of GR that were co-created by the applicant. Detailed aims of these publications can be summarized as follows:

- Initial introduction of novel extensions to GR [H1, H2, H4].

- Development of solutions to these models [H3, H5, H9]. The aim of this is to either a) use solutions to compare a model's predictions to observational constraints, or b) use solutions to characterize the health of the model - for example, explorations of whether pathological behavior exists in the model classically or quantum mechanically.
- Development of the canonical/Hamiltonian formulation of these models. This enables the deduction of important general results about the models as classical theories of gravity (for example the number of propagating degrees of freedom, stability properties), puts the theories' equations of motion in a form (Hamilton's equations of motion) which can aid the development of analytical and numerical solutions, and is a potentially important steps towards future attempts to quantize the models [H7, H8].
- Generalizations of the models and explorations of their theoretical basis [H6].

5.3 Summary

5.3.1 Type A models

Einstein's development of General Relativity heralded a remarkable result: that space and time are most conveniently considered as a unified object - spacetime - and that spacetime *is* the gravitational field. Einstein was able to make use of the mathematics of Riemannian geometry to describe spacetime as a four dimensional manifold on which is defined a (pseudo) Riemannian metric $g_{\mu\nu}$. At a point in spacetime, the metric gives information about local physical distances whereas derivatives of the metric enable the construction of a connection $\mathring{\Gamma}^{\rho}_{\mu\nu}(g,\partial g)$ which can be used to define covariant derivatives of tensor fields on the manifold. Finally, another tensor built from $\mathring{\Gamma}^{\rho}_{\mu\nu}(g,\partial g)$ and its derivatives - the Riemann curvature tensor $R^{\rho}{}_{\mu\nu\sigma}$ characterizes the 'non-flatness' of spacetime and the physical manifestation of this is the gravitational interaction that we observe.

This description of gravity has turned out to be enormously successful though with the aforementioned caveats mentioned in Section 5.1 regarding the need to introduce new forms of matter to explain the distribution and dynamics under gravity of known matter in the universe. If DM and cosmic inflation indicate a shortcoming in our understanding of the gravitational field then it may be that its description in terms of Riemannian geometry - and specifically it description entirely in terms of a metric tensor $g_{\mu\nu}$ - may be an approximation even within classical gravitation. We will argue that an alternative formulation of differential geometry due to the French mathematician Élie Cartan [E32] allows for a description of the gravitational field that extends GR to include degrees of freedom that may be lead to compelling phenomenology. In particular, we will focus on a model that may be relevant to the DM problem.

In Section 5.4.1 we provide a brief introduction to Cartan geometry and its use in the description of the gravitational field. In Section 5.4.2 we develop a set of models of gravitation based on Cartan-geometrical principles and provide a Hamiltonian/canonical analysis which demonstrates that a specific subset of the models yields an extension to GR with an additional, effective DM-type contribution to the gravitational field equations. In Section 5.4.3 these models are considered from an alternative perspective: that of models that arise from the 'gauging' of non-gravitational physical theories that nonetheless possess spacetime diffeomorphism symmetry. An overview of phenomenology of the models and scope for generalization of the model is given in Section 5.4.4.

Results in context: The surprising result that GR can be described as a spontaneously-broken gauge theory of the de Sitter ($SO(1,4)$) or anti de Sitter group ($SO(2,3)$) was discovered by MacDowell and Mansouri [E9], with substantial development of the idea shortly afterwards by Stelle and West [E13]. In this picture, gravity can be described by a gauge field/connection valued in the Lie algebra of one of these groups and a gravitational 'Higgs' field which breaks the symmetry down to the symmetry of the Lorentz group ($SO(1,3)$). These are to be regarded as 'internal' symmetries and independent of the spacetime diffeomorphism symmetry generally present in gravitational theory. The relation of the work of [E13] to Cartan's approach to geometry was investigated in detail in series article [H1].

Though the mathematical ingredients of approaches such as [E13] strongly resemble the gauge theories of particle physics, there was a crucial difference: it was assumed that the 'Higgs' field is non-dynamical/has no degrees of freedom and this has been the general approach in this area of research. This is of course is stark contrast to the Higgs field of the standard model. I feel that the most significant advance that I have made in this area is the discovery of a number of Cartan-geometrical models of gravity where this ‘Higgs’ field is a genuinely dynamical field and that this introduces compelling phenomenology that may be related to open problems in cosmology such as the nature of DM.

5.3.2 Type B models

The earliest evidence for the existence of DM came not from the very largest scales in the universe but from astrophysical systems such as clusters of galaxies where it was found that galaxies seemed to be under a gravitational influence greater than that which they were able to provide amongst themselves [E1, E2, E3]. A similar phenomenon on a smaller scale was later observed with the motion of stars within individual galaxies [E8, E11]. Milgrom proposed [E15, E16, E17] that this latter effect could, instead, result from modifying the inertia or dynamics of baryons or the gravitational law at accelerations smaller than $a_0 \sim 1.2 \times 10^{-10}\mathrm{m/s}^2$. The latter is further explored in [E18] where if gradients of the gravitational potential Φ are smaller than a_0, nonrelativistic gravity is effectively governed by

$$\vec{\nabla} \cdot \left(\frac{|\vec{\nabla}\Phi|}{a_0} \vec{\nabla}\Phi \right) = 4\pi G_{\mathrm{N}} \rho. \tag{1}$$

Here, G_{N} is the Newtonian gravitational constant, and ρ the matter density. These models are referred to as modified Newtonian dynamics (MOND).

Much work has gone into deducing astrophysical consequences of MOND, its consistency with data [E76]. It is inherently nonrelativistic and, thus, impossible to test in many regimes where GR is known to explain gravitational phenomenon as well as cosmological settings where systems such as the CMB require a relativistic treatment. Furthermore, in the absence of a fully relativistic ‘completion’, it is not known when (1) is expected to break down. This can be compared to the situation of the Newtonian limit of GR, where GR gives a precise account of when corrections to Newton’s theory become important. Therefore, to test the equation (1) it is vital to develop fully-relativistic theories whose field equations reduce to this form in some appropriate limit. The Type B models are specific examples of such theories. The models are notable in that - as will be shown - they are the first gravitational models that reduce to (1) in the appropriate limit whilst also being a successful alternative to DM on the largest, cosmic scales.

In Section 5.5.1, a general class of models will be considered, illustrating what conditions they must satisfy for gravitational waves in the model to propagate at the speed of light. This is an extremely important constraint on theories of modified gravity. In Section 5.5.2 a more specific set of models will be proposed (the AeST models) and it will be demonstrated that these are models that possess a MOND limit in the appropriate regime and that provide an account of large scale cosmological data as good as the standard cosmological model but without DM. The recovery of this result and the limit (1) within the same theoretical framework is arguably a significant achievement. In Section 5.5.3 the stability of Minkowski space in the AeST models is explored, and in Section 5.5.4 the full non-perturbative Hamiltonian formulation of the models is developed. Finally in Section 5.5.5, a number of further cosmological results involving the AeST models are briefly discussed.

Results in context: MOND was proposed as a non-relativistic modification to gravity in 1983 at a time when data regarding the large scale structure of the universe was very limited. In the following decades, a number of experiments (for example COBE [E29], WMAP [E40], and Planck [E111]) measured observables such as the pattern of anisotropies in the CMB with ever-increasing accuracy. The picture that emerged was that GR alongside significant contributions from DM and a cosmological constant Λ was an excellent match for the data. The fact that DM is an indispensable part of the standard model of cosmology led to a popular perception in the field that even if the modification to Newton’s law (1) can be embedded in a fully relativistic theory of gravity, it is unlikely that such a model could successfully describe results from precision cosmology. Independent of this, the observation that the speed of gravity is - to high accuracy- equal to the speed of light led to claims [E101] that relativistic completions of (1) would likely be in conflict with the data. The Type B models that are described in the series are a proof of concept that there do exist theories of gravity that have the phenomenology of MOND, gravitational waves that propagate at the speed of light, and that successfully describe cosmological data without the addition of DM.

5.4 Model A

5.4.1 Cartan geometry and gravitation

Riemannian geometry forms the mathematical basis of Einstein's General Relativity. The metric representation of Riemannian geometry consists of the pair of variables $\{g_{\mu\nu}, \Gamma^{\rho}_{\mu\nu}\}$. Whilst the metric tensor $g_{\mu\nu}$ encodes all information of distances between points on a manifold, the affine connection $\Gamma^{\rho}_{\mu\nu}$ encodes the information about parallel transport of tangent vectors u^{μ} as well as enable the construction of a covariant derivative ∇_{μ} acting on tensors. Conceptually $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ are independent objects. Within Riemannian geometry, however, the connection $\Gamma^{\rho}_{\mu\nu}$ is required to be metric-compatible and torsion-free:

- Metric compatibility: $\nabla_{\rho} g_{\mu\nu} \equiv \partial_{\rho} g_{\mu\nu} - \Gamma^{\sigma}_{\mu\rho} g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\rho} g_{\mu\sigma} = 0.$
- Zero torsion: $\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = 0$

The affine connection can then be uniquely determined from the metric

$$\mathring{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \Big(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \Big) \tag{2}$$

and it becomes natural to view the metric as the primary variable and the affine connection as a secondary, derived quantity.

An alternative approach due to Cartan [E32] is a generalization of an intuitive way of probing the geometry of a surface. Consider a curved two dimensional surface. If we want to know the length of a path between two points on that surface, we can imagine taking a ball and rolling it (without slipping or twisting) along the path. In this way we can connect a length on a potentially complicated curved surface to a set of infinitesimal rotations of the ball. Additionally, it can be shown [H1] that the intrinsic curvature of this surface at a point is in correspondence with how much a ball has rotated after being rolled (again without slipping or twisting) around a small loop beginning and ending at that point. By way of example, consider a two dimensional surface coordinatized by coordinates x^{μ} embedded in three dimensional Euclidean space $\mathbb{R}^3$ (coordinatized by Euclidean coordinates X^A and with Euclidean metric δ_{AB}). The ball placed on the surface at a point p_1 defines a vector $V^A(p_1)$ corresponding to a line of squared-length $\delta_{AB} V^A V^B$ [1]. If the ball is rolled to a nearby point p_2 then the contact between the ball and the surface defines a new vector

$$V^A(p_2) = V^A(p_1) + \partial_{\mu} V^A(p_1) \delta x^{\mu} + \dots \tag{3}$$

where dots denote terms of higher order in the small coordinate increments δx^{μ}. Meanwhile, the original vector $V^A(p_1)$ has been 'rolled' by the rotation of the ball and we can define this rolled vector $V_{|}(p_2)$ at point p_2 to be related to its original value at p_1 via

$$V_{|}^{A}(p_2) = V^A(p_1) - A^{A}{}_{B\mu}(p_1) V^B \delta x^{\mu} + \dots \tag{4}$$

Note that the rolling preserves the norm of the vector, hence $V_{A|}(p_2) V_{|}^{A}(p_2) = V_A(p_1) V^A(p_1)$ [2]. Then, using (4) it follows that:

$$A^{AB}{}_{\mu} = -A^{BA}{}_{\mu} \tag{5}$$

If the surface has a metric $g_{\mu\nu}$ then infinitesimal path joining p_1 and p_2 has squared-length $\delta s_g^2 = g_{\mu\nu} \delta x^{\mu} \delta x^{\nu}$. As the ball has been rolled without slipping, this should correspond to the squared infinitesimal arc length δs_b^2 subtended on the ball, where

[1] Here and throughout this document we will use the Einstein summation convention that repeated indices are understood to be summed over.

[2] Where indices are lowered and raised with δ_{AB} and δ^{AB} respectively.

$$\delta s_B^2 = \delta_{AB}(V^A(p_2) - V_|^A(p_2)(V^B(p_2) - V_|^B(p_2))$$
$$= \delta_{AB} D_\mu V^A D_\nu V^B \delta x^\mu \delta x^\nu \quad (6)$$
$$D_\mu V^A \equiv \partial_\mu V^A + A^A{}_{B\mu} V^B \quad (7)$$

The identification $\delta s_g^2 = \delta s_b^2$ partially fixes the symbols $A^A{}_{B\mu}$ - in this embedding picture it corresponds to rolling the ball without slipping. That the intrinsic curvature should be a measure how much the ball has been rotated when rolled around a path corresponds to rolling also without slipping and this fixes the remainder of $A^A{}_{B\mu}$ [H1].

Thus, the geometry of a surface can be characterized in terms of $\{g_{\mu\nu}(x), \Gamma^\rho_{\mu\nu}(x)\}$, or, equivalently, the variables $\{V^A(x), A^A{}_{B\mu}(x)\}$. The rolling process can be thought of as a rotation in 3d space that depends on the position x^μ on the surface i.e. a series of local $SO(3)$ transformations. Then, the role of $A^A{}_{B\mu}(x)$ is as a connection/gauge field, which can used to construct a covariant derivative $D_\mu V^A(x)$ acting on the vector $V^A(x)$ which distinguishes between parts of the connection $A^A{}_{B\mu}$ that contain information about the metric, and parts that contain information about curvature. Indeed one can dispense entirely with the embedding picture and construct a description of the geometry from $\{V^A(x), A^A{}_{B\mu}(x)\}$. A compelling feature of these variables is they have great similarity to the mathematical ingredients of the spontaneously-broken gauge theories of the standard model: $A^A{}_{B\mu}$ is a gauge field valued in the Lie algebra of $SO(3)$ whereas V^A is a 'gravitational' Higgs field whose non-vanishing vacuum expectation value spontaneously breaks the original symmetry (here $SO(3)$) to a smaller one ($SO(2)$).

For the case of the four dimensional spacetime of our universe, the transformations can be taken to be local $SO(1,4)$ (de Sitter group) or $SO(2,3)$ (anti de Sitter group) transformations where the field V^A, if it dynamically achieves a non-vanishing spacelike (de Sitter case) or timelike (anti de Sitter case) expectation value, the remnant symmetry in gravitation is the local $SO(1,3)$ (Lorentz group) symmetry of the Einstein-Cartan formulation of gravity. Actions for the gravitational field can be constructed which display this structure [E10, E14]. The novelty of the approach of [H1] was to consider V^A as a genuine dynamical field, in contrast to prior approaches such as [E14] that constrain V^A so that it has no dynamical degrees of freedom; arguably this procedure is artificial and would be analogous to constraining the electroweak Higgs field which would in essence remove the Higgs boson from the SM. It is shown in [H1] that Cartan-geometrical models of gravity with dynamical symmetry breaking of $SO(2,3)$ or $SO(1,4)$ gauge symmetry can lead to a swathe of novel extensions to GR. For example, a simple polynomial action in fields $\{V^A, A^A{}_{B\mu}\}$ contains the Peebles-Ratra scalar tensor theory [O15, E39] whilst an extension of this action allows for cosmological solutions displaying a classical change of metric signature from Euclidean to Lorentzian in the early universe [O14].

A remarkable feature of GR is its local spacetime diffeomorphism symmetry. Despite its elegant formulation as a spacetime theory where the basic variables are the spacetime metric $g_{\mu\nu}$ and spacetime connection $\Gamma^\rho_{\mu\nu}$, the theory may be cast in canonical/Hamiltonian form where the spacetime manifold is assumed to have topology $R \times \Sigma$ (and that the surface Σ can be coordinatized by coordinates x^a with R coordinatized by a coordinate time t). where the pullback of the spacetime metric to Σ is a spatial metric h_{ab} (with accompanying pullback γ^c_{ab} of $\Gamma^\rho_{\mu\nu}$ to Σ). It is the field h_{ab} and its momentum $\tilde{\pi}^{ab}$ that become the dynamical fields describing gravitation. We have seen how a rolling connection can contain the information about the intrinsic geometry $\{h_{ab}, \gamma^c_{ab}\}$ of a surface.

Can the notion of a Cartan connection be extended so as to additionally include information about such a surface is evolving with respect to a spacetime that it is a submanifold of i.e. can a rolling connection contain information about $\{h_{ab}, \gamma^c_{ab}, \tilde{\pi}^{ab}\}$? This is the focus of series publication [H2] and the result is a surprising connection between Cartan geometry, representation theory of the complexified Lorentz group, and an earlier formulation of gravitation due to Ashtekar [E19] which was found to greatly simplify the canonical formalism for gravity.

Instead of considering a two dimensional surface embedded in $\mathbb{R}^3$ we consider a three dimensional surface (coordinatized by x^a) embedded in $\mathbb{C}^{1,3}$ (coordinatized by Z^I) and instead of a ball, one considers 'rolling' an object with contact vector ϕ^I satisfying $\eta_{IJ}\phi^I\phi^J \equiv \phi^2 < 0 \in \mathbb{R}$. Then one can transport the object on the manifold according to a connection $\mathcal{A}^I{}_{Ja}$ which preserves the norm ϕ^2. The transformation of configuration of the object is therefore described by an element of the *complexified* Lorentz group $SO(1,3)_C$ and $\mathcal{A}^I{}_{Ja}$ is valued in its Lie algebra. Rolling in $\mathbb{C}^4$ generalizes the earlier $\mathbb{R}^3$ example in that the rolling can involve 'imaginary twisting' that creates a purely imaginary part to the rolled vector ϕ^I.

Recall that an object F^{IJ} valued in the Lie algebra of $SO(1,3)_C$ (i.e. $F^{IJ} = -F^{JI}$) one can decompose the form into *self-dual* (+) and *anti self-dual* (-) parts as $F^{IJ} = F^{+IJ} + F^{-IJ}$, $F^{\pm IJ} = \frac{1}{2}(F^{IJ} \mp i\epsilon^{IJ}{}_{KL}F^{KL}/2)$ where $\epsilon^{IJ}{}_{KL}F^{\pm KL} = \pm 2iF^{\pm IJ}$ [3]. Hence $\mathcal{A}^{IJ}{}_{\mu}$ can be decomposed into self-dual and anti self-dual parts $\mathcal{A}^{\pm IJ}{}_{a}$. It is shown in [H2] that the metric h_{ab} on the surface can be associated with

$$h_{ab} = \eta_{IJ} D_a\phi^I D_b\phi^J \tag{8}$$

where $D_a\phi^I \equiv \partial_a\phi^I + \mathcal{A}^I{}_{Ja}\phi^J$. Furthermore, we identify $\mathcal{A}_a^{+IJ}$ with a complex combination of the Christoffel symbols γ^c_{ab} associated with h_{ab} and the extrinsic curvature K_{ab} of the surface as if it were embedded in a curved spacetime with metric $g_{\mu\nu}$. This specific combination is precisely the pullback of the self-dual Ashtekar connection to spatial hypersurfaces [E19]. Thus, one a three dimensional surface, the connection $\mathcal{A}^I{}_{Ja}$ can be constructed to contain information about distances in the surface, the intrinsic curvature of the surface, *and* the extrinsic curvature of the surface as it were embedded in a specific curved four dimensional spacetime. Remarkably, it was found in [H2] that there exists a simple polynomial action for fields $\{\phi^I, \mathcal{A}^I{}_{J\mu}$ which gives rise to field equations with dynamical solutions where the pullback $\mathcal{A}^I{}_{Ja}$ of the spacetime Cartan connection $\mathcal{A}^I{}_{J\mu}\}$ to surfaces of constant ϕ^2 yielded precisely the content of $\mathcal{A}^I{}_{Ja}$ described in the construction above. Furthermore, a preliminary analysis suggested, importantly, this action principle possessed a General Relativistic limit (corresponding specifically to Ashtekar's chiral formulation of GR) whilst additionally possessing solutions that represent an effective DM contribution to the Einstein equations. A detailed analysis of this action in Hamiltonian form will be discussed in detail Section 5.4.2 and in the action in Lagrangian form will be discussed in detail in Section 5.4.3.

5.4.2 Hamiltonian formulation of gravity as a spontaneously-broken gauge theory of the Lorentz group

Maintaining general spacetime covariance is a typical requirement in gravitational theories. Dynamical fields are spacetime tensors and actions are coordinate-independent functionals built entirely from these fields. As such, there is a-priori no preferred notion of time in gravitational theory. However, significant insights can be gained into the structure of generally covariant theories by choosing a single coordinate by which to measure the change of fields over the spacetime manifold. This is the 3+1 decomposition of physical theories and enables the canonical/Hamiltonian analysis of the theory. Canonical/Hamiltonian analysis of a classical field theory is a very valuable tool. It allows for the determination of the number of degrees of freedom that a theory has, and the casting of the Euler-Lagrange equations as a set of differential equations which are first order in time - via Hamilton's equations of motion - is typically very valuable for the determination of solutions to the theory, be it via analytical or numerical methods. Furthermore, by its very nature, the casting of a theory in canonical form opens the path to canonical quantization of the theory. The canonical formulation of GR has also served as a starting point for alternative non-perturbative quantization methods such as loop quantum gravity [E23].

The starting point is a generalization of the action considered in [H2]:

$$S[\phi^I, \omega_\mu^{+IJ}, \omega_\mu^{-IJ}] = \int d^4x\, \tilde{\epsilon}^{\mu\nu\alpha\beta}\epsilon_{IJKL}D_\mu\phi^I D_\nu\phi^J \left(g_+ R^{KL}{}_{\alpha\beta}(\omega^+) + g_- R^{KL}{}_{\alpha\beta}(\omega^-)\right) \tag{9}$$

We assume that the spacetime manifold M is topologically $R\times\Sigma$, where Σ is a three-dimensional submanifold of M and R may be coordinatized by a number t which can be thought of as 'coordinate time'. However, we emphasize that t is to be regarded as a number labeling different submanifolds of M and may not be straightforwardly related to proper time as determined by a spacetime metric (indeed we will encounter solutions of the models (9) where no notion of proper time exists). Furthermore, we are primarily interested in solutions to the theory in the bulk spacetime and will neglect boundary conditions and surface integrals. Hence our treatment will be exact only in cases where Σ is a closed manifold [E27]. Alongside the label t for coordinate time, we will use a set of three coordinates $\{x^a\}$ $(a = 1,2,3)$ to cover the manifold Σ. In accordance with the 3+1 decomposition, we introduce the following decomposition of the fields $\omega_\mu^{\pm IJ}$:

[3] We use the convention $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$ and introduce the Levi Civita symbol $\epsilon_{IJKL} = \epsilon_{[IJKL]}$.

$$\omega^{\pm IJ}{}_{\mu}dx^{\mu} = \Omega^{\pm IJ}dt + \beta^{\pm IJ}{}_{a}dx^{a} \tag{10}$$

Furthermore, for notational compactness, we introduce the following quantities:

$$R^{\pm IJ}_{ab} \equiv 2\left(\partial_{[a}\beta^{\pm IJ}{}_{b]} + \beta^{\pm IK}{}_{[a|}\beta^{\pm J}_{K}{}_{|b]}\right) \tag{11}$$

$$e^I_a \equiv \partial_a\phi^I + \beta^I{}_{Ja}\phi^J \tag{12}$$

The Lagrangian density for the action (9) can be cast into the following form which yields the same field equations as the original action:

$$\tilde{\mathcal{L}}(\phi^I, \tilde{P}_I, \beta^{\pm IJ}_a, \tilde{P}^{a\pm}_{IJ}, \Omega^{\pm IJ}, V^I, V^{\pm IJ}_a) = \tilde{P}_I\dot{\phi}^I + \tilde{P}^{+c}_{IJ}\dot{\beta}^{+IJ}{}_c + \tilde{P}^{-c}_{IJ}\dot{\beta}^{-IJ}{}_c - \tilde{\mathcal{H}} + \partial_a\tilde{\ell}^a \tag{13}$$

Where

$$\tilde{\mathcal{H}} = -\Omega^{+IJ}\tilde{\mathcal{G}}^+_{IJ} - \Omega^{-IJ}\tilde{\mathcal{G}}^-_{IJ} + V^I\tilde{C}_I + V^{+IJ}_a\tilde{C}^{+a}_{IJ} + V^{-IJ}_a\tilde{C}^{-a}_{IJ} \tag{14}$$

$$\tilde{\mathcal{G}}^+_{IJ} \equiv D^{(\beta^+)}_c\tilde{P}^{+c}_{IJ} + \left[\tilde{P}_{[I}\phi_{J]}\right]^+ \tag{15}$$

$$\tilde{\mathcal{G}}^-_{IJ} \equiv D^{(\beta^-)}_c\tilde{P}^{-c}_{IJ} + \left[\tilde{P}_{[I}\phi_{J]}\right]^- \tag{16}$$

$$\tilde{C}_I = \tilde{P}_I - 2g_+\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e^J_aR^{+KL}_{bc} - 2g_-\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e^J_aR^{-KL}_{bc} \tag{17}$$

$$\tilde{C}^{+c}_{IJ} = \tilde{P}^{+c}_{IJ} - 2g_+\left[\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e^K_ae^L_b\right]^+ \tag{18}$$

$$\tilde{C}^{-c}_{IJ} = \tilde{P}^{-c}_{IJ} - 2g_-\left[\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e^K_ae^L_b\right]^- \tag{19}$$

where we've introduced the 'spatial covariant derivative' with respect to any of $\mathcal{B} = \{\beta, \beta^+, \beta^-\}$:

$$\begin{aligned} D^{(\mathcal{B})}_aY^{AB\ldots}{}_{CD\ldots} \equiv{}& \partial_aY^{AB\ldots}{}_{CD\ldots} + \mathcal{B}^A{}_{Ea}Y^{EB\ldots}{}_{CD\ldots} + \mathcal{B}^B{}_{Ea}Y^{AE\ldots}{}_{CD\ldots} + \ldots \\ & - \mathcal{B}^E{}_{Ca}Y^{AB\ldots}{}_{ED\ldots} - \mathcal{B}^E{}_{Da}Y^{AB\ldots}{}_{CE\ldots} + \ldots \end{aligned} \tag{20}$$

The total derivative $\partial_a\tilde{\ell}^a$ will be neglected as an ignorable boundary term. The Lagrangian density (13) is in canonical form, with phase space being coordinatized by fields $(\phi^I, \beta^{\pm IJ}_a)$, their momenta $(\tilde{P}_a, \tilde{P}^{a\pm}_{IJ})$ subject to constraints (15)-(19) which are enforced by stationarity of the action with respect to the Lagrange multiplier fields $(\Omega^{\pm IJ}, V^I, V^{\pm IJ}_a)$.

The Hamiltonian analysis then proceeds by calculating the time evolution of constraints using the Euler-Lagrange equations following from (13). It is to be required that the constraints are preserved by time evolution and within the equations of motion this either fixes Lagrange multiplier fields in terms of the phase space fields or leads to new constraints amongst the phase space variables, the preservation of which under time evolution must further be checked until no new constraints are generated. Upon completion of this constraint analysis, the final set of constraints can be grouped into two categories: *first class constraints*: these are constraints that commute with all other constraints according to the unconstrained phase space Poisson bracket; *second class constraints:* these are constraints that do not commute with all other constraints. It is found that the constraint structure crucially depends on the value of the constants (g_+, g_-). If $g_+ \neq g_-$ the theory has constraint structure shown in Figure 1. If $g_+ = g_-$, the theory has the constraint structure shown in Figure 2.

For the general case $g_+ \neq g_-$, the classification of constraints is illustrated in Figure 1. Given the classification of constraints, we can now count how many complex degrees of freedom the model possesses. The dimensionality of the phase space per spatial point is

$$P = 8\,(\phi^I, \tilde{P}_I) + 18\,(\beta^{+IJ}_a, \tilde{P}^{+a}_{IJ}) + 18\,(\beta^{-IJ}_a, \tilde{P}^{-a}_{IJ}) = 44$$

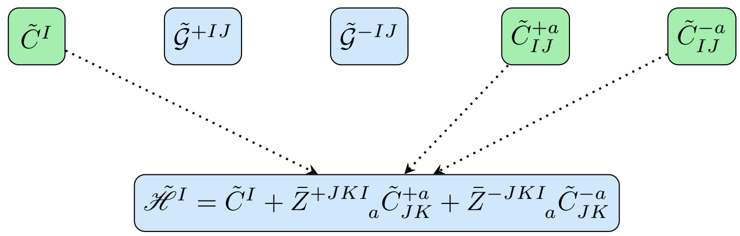

Figure 1: The structure of constraints in the case $g_+ \neq g_-$. First-class constraints are blue whilst constraints that are individually second-class are shown as green. The constraint analysis reveals that a linear combination of second-class constraints yields the first-class constraints $\tilde{\mathscr{H}}^I$. Symbols $\bar{Z}^{\pm JKI}{}_a$ are built from the phase space fields and are defined in [H8].

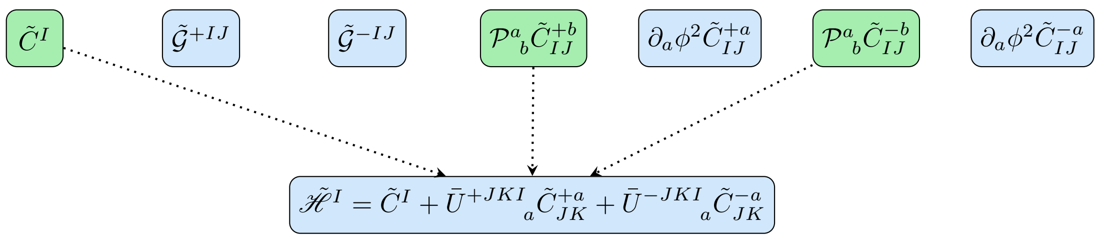

Figure 2: The structure of constraints in the case $g_+ = g_-$. First class constraints are blue whilst second class constraints are green. Unlike in the case $g_+ \neq g_-$, a subset of the individual $\tilde{C}^{\pm a}_{IJ}$ constraints are first class. As in the $g_+ \neq g_-$ constraint analysis reveals that a linear combination of individually second class constraints yields the first class constraints $\tilde{\mathscr{H}}^I$. Symbols $\bar{U}^{\pm JKI}{}_a$ are built from the phase space fields and are defined in [H8].

The number of first-class constraints is

$$F = 3\,(\tilde{G}^{+IJ}) + 3\,(\tilde{G}^{-IJ}) + 4\,(\tilde{\mathscr{H}}^I) = 10$$

and the number of second-class constraints is

$$S = 9\,(\tilde{C}^{+a}_{IJ}) + 9\,(\tilde{C}^{-a}_{IJ}) = 18$$

The number of degrees of freedom per spatial point is therefore

$$\begin{aligned} DOF &= \frac{1}{2}(P - 2F - S) \\ &= 3 \end{aligned} \tag{21}$$

For the special case $g_+ = g_-$, the classification of constraints is illustrated in Figure 2. As in the previous case, the dimensionality of the phase space per spatial point is

$$P = 8\,(\phi^I, \tilde{P}_I) + 18\,(\beta^{+IJ}_a, \tilde{P}^{+a}_{IJ}) + 18\,(\beta^{-IJ}_a, \tilde{P}^{-a}_{IJ}) = 44$$

However, now the number of first-class constraints is

$$F = 3\,(\tilde{G}^{+IJ}) + 3\,(\tilde{G}^{-IJ}) + 4\,(\tilde{\mathscr{H}}^I) + 3(\partial_a \phi^2 \tilde{C}^{+a}_{IJ}) + 3\,(\partial_a \phi^2 \tilde{C}^{-a}_{IJ}) = 16$$

and the number of second-class constraints is

$$S = 6\,(\mathcal{P}^a{}_b \tilde{C}^{+b}_{IJ}) + 6\,(\mathcal{P}^a{}_b \tilde{C}^{-b}_{IJ}) = 12$$

The number of degrees of freedom per spatial point is therefore

$$\begin{aligned} DOF &= \frac{1}{2}(P - 2F - S) \\ &= 0 \end{aligned} \tag{22}$$

Therefore the theory with $g_+ = g_-$ possesses no degrees of freedom. Indeed, this case has more first-class constraints than the case $g_+ \neq g_-$ and so it is to be expected that this specific case has more symmetry than the general case. Indeed, the action (9) in this case possesses a symmetry under the transformation:

$$\phi^I \to \phi^I, \quad \omega_\mu^{IJ\pm} \to \omega_\mu^{IJ\pm} + \partial_\mu \phi^2 \xi^{\pm IJ} \tag{23}$$

By examining the propagation of linear metric perturbations on Minkowski space, it can be shown that only for the cases $(g_+ = 1, g_- = 0)$ and $(g_- = 0, g_+ = 1)$ is the expression of GR recovered [H8]. We will restrict ourselves to these two cases and first focus on the former case and later demonstrate how the latter case can be straightforwardly recovered. When $(g_+ = 1, g_- = 0)$, the primary Hamiltonian density $\tilde{\mathcal{H}}$ simplifies considerably. Recalling its general form

$$\tilde{\mathcal{H}} = -\Omega^{+IJ}\tilde{\mathcal{G}}^+_{IJ} - \Omega^{-IJ}\tilde{\mathcal{G}}^-_{IJ} + V^I\tilde{C}_I + V_a^{+IJ}\tilde{C}^{+a}_{IJ} + V_a^{-IJ}\tilde{C}^{-a}_{IJ}, \tag{24}$$

the constraints now simplify to:

$$\tilde{\mathcal{G}}^+_{IJ} \equiv D_c^{(\beta^+)}\tilde{P}^{+c}_{IJ} + \left[\tilde{P}_{[I}\phi_{J]}\right]^+ \tag{25}$$

$$\tilde{\mathcal{G}}^-_{IJ} \equiv D_c^{(\beta^-)}\tilde{P}^{-c}_{IJ} + \left[\tilde{P}_{[I}\phi_{J]}\right]^- \tag{26}$$

$$\tilde{C}_I = \tilde{P}_I - 2\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e_a^J R_{bc}^{+KL} \tag{27}$$

$$\tilde{C}^{+c}_{IJ} = \tilde{P}^{+c}_{IJ} - 2\left[\epsilon_{IJKL}\tilde{\varepsilon}^{abc}e_a^K e_b^L\right]^+ \tag{28}$$

$$\tilde{C}^{-c}_{IJ} = \tilde{P}^{-c}_{IJ} \tag{29}$$

Given these simplifications, it is possible to algebraically solve for the $(\beta_a^{-IJ}, \tilde{P}^{-a}_{IJ})$ and eliminate them from the variational problem. Firstly, from (29) we have $\tilde{P}^{-a}_{IJ} = 0$. Then, recalling the definition (12), the constraint $\tilde{C}^{+c}_{IJ}$ can be regarded as an equation for which one can solve for $\beta_a^{-IJ} = \beta_a^{-IJ}(\beta^+, \tilde{P}^+, \phi, \partial\phi)$. Therefore the second-class constraints can be solved. Given these solutions, the constraint $\tilde{\mathcal{G}}^-_{IJ}$ simplifies to $\left[\tilde{P}_{[I}\phi_{J]}\right]^- = 0$; if we decompose $\tilde{P}^I = \tilde{\Pi}\phi^I + \tilde{P}^I_\perp$, where $\tilde{P}^I_\perp\phi_I = 0$, then $\tilde{\mathcal{G}}^-_{IJ} = 0$ can be taken to imply the solution $\tilde{P}^I_\perp = 0$ which we now adopt. Additionally the quantity $V^I\tilde{C}_I$ can be expressed in terms of Arnowitt-Deser-Misner densitized lapse and shift functions $(\underset{\sim}{N} \equiv N/\sqrt{q}, N^a)$ [E27] and the remaining phase space variables so that the gravitational Lagrangian density can be written:

$$\begin{aligned} \tilde{\mathcal{L}}[\tilde{P}^{+a}_{IJ}, \beta_a^{+IJ}, \tilde{\Pi}, \phi^2, \Omega^{+IJ}, \underset{\sim}{N}, N^a] = {}& \frac{1}{2}\tilde{\Pi}\dot{\phi}^2 + \tilde{P}^{+a}_{IJ}\dot{\beta}^{+IJ}{}_a + \Omega^{+IJ}\left(D_a^{(\beta^+)}\tilde{P}^{+a}_{IJ}\right) \\ & - \underset{\sim}{N}\left(\xi\tilde{\Pi}\sqrt{q}\sqrt{-\phi^2 + \frac{1}{4}q^{ab}\partial_a\phi^2\partial_b\phi^2} - \frac{1}{4}\tilde{P}^{+a}_{IK}\tilde{P}^{+bK}{}_J R^{+IJ}{}_{ab}\right) \\ & - N^a\left(\frac{1}{2}\tilde{\Pi}\partial_a\phi^2 + \tilde{P}^{+b}_{IJ}F^{+IJ}_{ab} - \beta_a^{+IJ}D_b^{(\beta^+)}\tilde{P}^{+b}_{IJ}\right) \end{aligned} \tag{30}$$

where $\sqrt{q}$ and q^{ab} has been expressed in terms of $\tilde{P}^{+a}_{IJ}$ [H8] and we have redefined $\Omega^{+IJ} \to \Omega^{+IJ} + N^a\beta_a^{+IJ}$ so that the constraint obtained from the N^a equation of motion when smeared with a field ζ^a generates (non-Lorentz covariant) spatial diffeomorphisms $f \to f + \mathcal{L}_\zeta f$ on fields f in the phase space coordinatized by $(\phi^2, \tilde{\Pi}, \beta_a^{+IJ}, \tilde{P}^{+a}_{IJ})$. The Lagrangian density that we have recovered corresponds to the canonical formulation of Ashtekar's theory of gravity [E27] coupled to a field $\phi^2 = \eta_{IJ}\phi^I\phi^J$ whose dynamics is classically that of a pressureless perfect fluid when $\phi^2 < 0$ [E77, H6]. If $\phi^2 < 0$ and local coordinates are selected so that $\partial_a\phi^2$ for some region of spacetime then the energy density of the fluid is of sign $\xi\tilde{\Pi}$ and so the choice of the sign of $\xi = \mp 1$ reflects the relative sign of the energy density and $\tilde{\Pi}$. More generally, ϕ^2 by definition is not positive-definite and its equation of motion is:

$$\frac{1}{2}\partial_t \phi^2 = N\xi\sqrt{-\phi^2 + \frac{1}{4}q^{ab}\partial_a\phi^2\partial_b\phi^2} + \frac{1}{2}N^a\partial_a\phi^2 \tag{31}$$

The equation of motion for ϕ^2 differs from the equation of motion for, for example, the Higgs boson of the standard model; unlike that case, $\partial_t\phi^2$ is independent of the field's momentum $\tilde{\Pi}$ and generally the right-hand side of (31) will be non-zero whenever the spacetime metric is non-degenerate meaning that generally the magnitude of ϕ^2 will vary throughout spacetime. Notably there do exist solutions to the theory's field equations where, furthermore, the sign of ϕ^2 varies throughout spacetime [H6]. If instead, we had chosen the parameters $(g_+ = 0, g_- = 1)$ we would have instead recovered the *anti*-self-dual formulation of Ashtekar's theory coupled to an effective matter component described by $(\phi^2, \tilde{\Pi})$, the Lagrangian density of which can be recovered from (30) by the replacement of $(\beta_a^{+IJ}, \tilde{P}_{IJ}^{+a})$ with $(\beta_a^{-IJ}, \tilde{P}_{IJ}^{-a})$. Can the degree of freedom $\phi^2 = \eta_{IJ}\phi^I\phi^J$ act as a useful 'clock field' in physics? If we choose initial data such that $\partial_a\phi^2|_{t=t_0} = 0$ then the equation of motion for ϕ^2 at the initial moment becomes:

$$\frac{1}{2}\partial_t\phi^2 = \xi N\sqrt{-\phi^2} \tag{32}$$

The condition that $\partial_a\phi = 0$ will be preserved if $N = N(t)$. Given this condition, we can regard the integration of equation (32) as providing a functional relation between ϕ^2 and t. If $N = 1/\xi$ (which implies that t corresponds to proper time [H8]) then $\phi^2 = -t^2$ is a solution. If instead $N = 1/(2\xi\sqrt{t})$ then $\phi^2 = -t$ is a solution. Therefore we will associate a choice of $N(t)$ with a partial spacetime gauge fixing (N^a is left undetermined) and denote the associated time coordinate as t_ϕ. We may additionally solve the Hamiltonian constraint for $\tilde{\Pi}$ in this gauge to yield the following Lagrangian density:

$$\tilde{\mathcal{L}}[\tilde{P}_{IJ}^{+a}, \beta_a^{+IJ}, \Omega^{+IJ}, N^a] \stackrel{*}{=} \tilde{P}_{IJ}^{+a}\frac{\partial}{\partial t_\phi}\beta^{+IJ}{}_a - \mathcal{H}_{Phys} + \Omega^{+IJ}\left(D_a^{(\beta^+)}\tilde{P}_{IJ}^{+a}\right) - N^a\left(\tilde{P}_{IJ}^{+b}F_{ab}^{+IJ} - \beta_a^{+IJ}D_b^{(\beta^+)}\tilde{P}_{IJ}^{+b}\right) \tag{33}$$

$$\mathcal{H}_{Phys} = -\frac{N(t_\phi)}{4\sqrt{q}}\tilde{P}_{IK}^{+a}\tilde{P}^{+bK}{}_J R^{+IJ}{}_{ab} \tag{34}$$

Therefore we recover an action principle for gravity with a phase space coordinatized by fields $(\tilde{P}_{IJ}^{+a}, \beta_a^{+IJ})$, with a physical Hamiltonian density $\mathcal{H}_{Phys}$ and phase space constraints implemented by stationarity of the action under variation of (Ω^{+IJ}, N^a). Note that now N appears not as an independent field but as a fixed function of time and only in gauges $\partial_{t_\phi}N(t_\phi) = 0$ does the physical Hamiltonian not have explicit time-dependence (as opposed to the implicit time-dependence it possesses via its dependence on time-varying phase space fields), which corresponds to the case where $\sqrt{-\phi^2}$ measures metric proper time. The extension to GR is encoded in the fact that $\mathcal{H}_{Phys}$ is not constrained to vanish. Non-vanishing values would be interpreted as an effective energy density of an additional dust-like gravitating component in physics [H6].

The presence of degrees of freedom which would classically correspond to a pressureless perfect fluid have been proposed as solutions to the 'problem of time' in quantum gravity [E24, E26, E28, E77]. In particular in [E77] a pressureless perfect fluid Lagrangian described by phase space fields $(T(x^a), P_T(x^2))$ was proposed, with it argued that the canonical gauge choice $T = t$ may be imposed prior to quantization. Notably, the classical equations of motion generally do not allow T to be used as a global time variable due to the generic formation of caustics on surfaces of constant T [E92]. This suggests that *if* the flow of time T in the quantum theory remains unimpeded, then quantum corrections to the classical equations of motion may become important near would-be caustic formation. An interesting illustration of this is how the classical Big Bang singularity of a dust-dominated universe can be replaced by an effective bounce when dust-time is used in a quantum cosmological description of the system [E114]. If degrees of freedom described by fields $(\phi^I, \tilde{P}_I)$ or $(T, \tilde{P}_T)$ are to play the role of DM, then these corrections may have an experimental signature via how they affect the distribution and evolution of DM density. The phenomenology of Model B is discussed in more detail in Section 5.4.4.

5.4.3 Gravitation via the gauging of parameterized field theories

The model (9) can be interpreted from the perspective of Cartan gravity. An alternative perspective is in terms of the gauging of non-gravitational physical theories so as to recover the gravitational interaction, which is the focus of series publication [H6].

The notion of gauge symmetry is a crucial part of the mathematical structure of the standard model of particle physics. Consider an action $S_\chi[\chi]$ describing the dynamics of a matter field χ that is invariant under a global (i.e. independent of location in spacetime) continuous symmetry represented by the transformation $\chi \to U\chi$ (where indices are suppressed for notational compactness). Typically the action will not be invariant under local symmetry transformations ($U = U(x^\mu)$) as derivative terms $\partial_\mu \chi$ present in the Lagrangian then do not transform homogeneously under this transformation. However, the global symmetry can generally be promoted to a local one by the introduction of an additional field A_μ - called a gauge field or connection - which allows for the creation of a *covariant derivative* which - if the transformation U can be represented as a matrix and χ belongs to the fundamental representation of the symmetry group - takes the form:

$$\partial_\mu \chi \to D^{(A)}_\mu \chi \equiv \partial_\mu \chi + A_\mu \chi \tag{35}$$

If, under the U transformation, $A_\mu \to U A_\mu U^{-1} - \partial_\mu U U^{-1}$ then $D^{(A)}_\mu \chi \to U D^{(A)}_\mu \chi$. The extension of the definition of $D^{(A)}_\mu$ to matter fields in other representations of the symmetry group is straightforward. Alongside the modification $S_\chi[\chi] \to S_\chi[\chi, A_\mu]$, the process is then completed by the introduction of an action $S_A[A_\mu, \chi]$ which allows for the dynamics of A_μ to be well defined. An example of this process would be that of a complex scalar field theory, where the Lagrangian density in inertial coordinates in Minkowski spacetime is $\mathcal{L}_\phi = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V(\phi^*\phi)$ is invariant under global $U(1)$ transformations $\phi \to e^{i\alpha}\phi$. The $U(1)$ invariance can be made local by introducing a field A_μ to construct the covariant derivative $D^{(A)}_\mu \phi = \partial_\mu\phi + A_\mu\phi$; the resultant locally $U(1)$ invariant action for ϕ is then supplemented by the Lagrangian density $\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, which provides dynamics for the field A_μ.

We will consider a similar gauging process in the context of gravitation, first reviewing existing results that show how it can be used to recover the model (9). To recap, special relativistic theories are commonly formulated in terms of matter fields existing in a space with fixed-geometrical structure i.e. Minkowski space and its accompanying metric tensor $\eta_{\mu\nu}$. An alternative approach is to not assume the presence of $\eta_{\mu\nu}$ but rather introduce a set of four scalar fields $X^I(x^\mu)$ which are dynamical in the sense the action is stationary with respect to small variations of these fields and this results in them having own equations of motion. Actions can be constructed so that dynamically the fields end up configured to play the role of inertial coordinate fields in spacetime with an effective metric emerging via the combination $\tilde{\eta}_{\mu\nu} = \eta_{IJ}\partial_\mu X^I \partial_\nu X^J$, where $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$. Such theories are referred to as 'parameterized field theories'. An interesting property of these theories is that the absence of fixed geometrical structure (such as $\eta_{\mu\nu}$) means that such actions possess a symmetry with respect to spacetime diffeomorphisms in the manner familiar from gravitational theory. In addition, the actions for parameterized field theories possess a global symmetry:

$$X^I \to \Lambda^I{}_J X^J + P^I \tag{36}$$

The combined effect of the orthogonal matrix $\Lambda^I{}_J$ (representing a Lorentz transformation of X^I) with P^I is that of a global Poincaré transformation of X^I, analogous to the global coordinate transformations that preserve $\eta_{\mu\nu} = (-1, 1, 1, 1)$ i.e. that preserve the form of the Minkowski metric in inertial coordinates.

For concreteness, consider the case of the electromagnetic field. The following action yields Maxwell's equations upon small variations of the field A_μ:

$$S[A_\mu] = -\frac{1}{4}\int d^4x \sqrt{-\det[\eta]}\,\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \tag{37}$$

where $\eta_{\mu\nu}$ is the non-dynamical metric tensor of Minkowski spacetime, $F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$, and $\{x^\mu\}$ are some set of coordinates describing points in spacetime (not necessarily Minkowski coordinates). Due to the fixed, flat geometry of spacetime there exist 'inertial' coordinate systems coordinatized by $\{X^I\}$, for which in a general coordinate system $\{x^\mu\}$

$$\eta_{\mu\nu} = \eta_{IJ}\frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu} \tag{38}$$

where $\eta_{IJ} = \text{diag}(-1,1,1,1)$. One might imagine looking to recover metric structure from $\eta_{IJ}\frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu}$ instead of the fixed background metric $\eta_{\mu\nu}$ and promote the fields X^I to being dynamical. Then, consider the following action:

$$S[A_\mu, X^I] = -\frac{1}{4}\int d^4x\sqrt{-\det[\tilde{\eta}]}\tilde{\eta}^{\mu\alpha}\tilde{\eta}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \tag{39}$$

where

$$\tilde{\eta}_{\mu\nu} \equiv \eta_{IJ}\frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu} \tag{40}$$

and where $\eta_{IJ} = \text{diag}(-1,1,1,1)$ and $\tilde{\eta}^{\mu\nu}$ is the matrix inverse of $\tilde{\eta}_{\mu\nu}$, which is assumed to exist. The action (39) is manifestly invariant under the global transformation (36) and the equation of motion for X^I can be shown to be

$$0 = \partial_\mu\left(\sqrt{-\det[\tilde{\eta}]}\frac{\partial X_I}{\partial x^\nu}T^{\mu\nu}\right) \tag{41}$$

where $T_{\mu\nu}$ is the stress energy tensor of the electromagnetic field; therefore the equation of motion for X^I expresses conservation of the stress energy tensor. There exist solutions $X^I(x^\mu)$ for which there exist coordinates such that $\partial X^I/\partial x^\mu \stackrel{*}{=} \delta^I_\mu$. In these coordinates, $\tilde{\eta}_{\mu\nu} = \text{diag}(-1,1,1,1)$ and so we see that X^I here play the role of inertial coordinates in Minkowski spacetime. As expected, for these solutions, this form of $\tilde{\eta}_{\mu\nu}$ is preserved by the transformation (36). Generally, the recovery of familiar classical field theory and quantum theory in Minkowski space is possible in the parameterized approach, though interestingly the latter makes use of techniques originating in the loop quantum gravity research program [E55].

The standard route to gravitation has been via Einstein's GR where $\eta_{\mu\nu}$ is promoted to a dynamical field (denoted $g_{\mu\nu}$) with its own action which is given - up to the necessary Gibbons–Hawking–York boundary term - by the Einstein-Hilbert action:

$$S_g[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R(g,\partial g) \tag{42}$$

where R is the Ricci formed from the Levi-Civita connnection and G is Newton's gravitational constant. This approach must be modified somewhat when fermionic fields are present. The actions of the standard model of particle physics consists of the following dynamical fields: gauge fields A_μ (spacetime one-forms), the electroweak Higgs field ϕ (a spacetime scalar), and fermionic fields Ψ_A and $\chi^{A'}$ (Weyl spinors i.e. spacetime scalars in the fundamental representations of $SL(2,C)$) alongside the non-dynamical object $\bar{e}^I_\mu$ (a spacetime one form in the fundamental representation of $SO(1,3)$ such that $\eta_{IJ}\bar{e}^I_\mu\bar{e}^J_\nu \equiv \eta_{\mu\nu}$). These actions are invariant under global $SL(2,C)$ transformations which act only on the Weyl spinors and on $\bar{e}^I_\mu$ via the group homomorphism between $SL(2,C)$ and $SO(1,3)$. The Lagrangian four forms $\mathcal{L}$ that are integrated to produce the actions of the standard model transform as differential forms under diffeomorphisms that act on *both* dynamical and non-dynamical fields but also transform as forms under diffeomorphisms that act *only* on the dynamical fields for diffeomorphisms generated by vector fields ξ^μ that satisfy $\pounds_\xi\eta_{\mu\nu} = 0$ (where $\pounds_\xi$ denotes the Lie derivative) i.e. ξ^μ that satisfy this equation are the Killing vectors of Minkowski space. There are ten independent $\xi^{(i)\mu}$ and their commutator $[\xi^{(i)},\xi^{(j)}]$ satisfies the Lie algebra of the Poincaré group $ISO(1,3)$. In this sense the actions of the standard model possess a global $SL(2,C)$ symmetry and the Lagrangian forms exhibit a global $ISO(1,3)$ covariance. It was shown by Kibble [E6] (building on earlier work by Utiyama [E4]) that the global $SL(2,C)$ symmetry could be promoted to a local one by the introduction of a gauge field ω valued in the Lie algebra of $SL(2,C)$ - such that the covariant derivative

$D^{(\omega)}_\mu \chi^{A'} \equiv \partial_\mu \chi^{A'} + \omega^{A'}{}_{B'\mu}\chi^{B'}$, where $\omega^{A'}{}_{B'\mu} = \frac{1}{8}\omega_{IJ\mu}(\bar{\sigma}^I\sigma^J - \bar{\sigma}^J\sigma^I)^{A'}{}_{B'}$, transforms homogeneously under this transformation. Additionally, the remaining presence of non-dynamical, prior geometry was removed by the introduction of a dynamical field e^I_μ to appear in place of $\bar{e}^I_\mu$ - such that $\eta_{IJ}e^I_\mu e^J_\nu \equiv g_{\mu\nu}$. The introduction of the set of dynamical fields $\{e^I_\mu, \omega^{A'}{}_{B'\mu}\}$ into the matter actions suggests that the gauging process should be completed by providing action allowing for a consistent dynamics of these degrees of freedom i.e. the introduction of gravitation as a dynamical interaction. A simple possibility is the following action:

$$S_g[\omega, e] = \frac{1}{64\pi G}\int d^4x \varepsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e^I_\mu e^J_\nu R^{KL}{}_{\alpha\beta}(\omega) \tag{43}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita *density* and $R^{IJ}{}_{\mu\nu} \equiv 2\partial_{[\alpha}\omega^{IJ}{}_{\beta]} + 2\omega^I{}_{K[\alpha}\omega^{KJ}{}_{\beta]}$. The action (43) is the Palatini action in the Einstein-Cartan formulation of gravity and its equations of motion are classically equivalent to GR with an additional matter term quadratic in fermionic currents.

This procedure can be interpreted as a combined gauging of internal symmetries and the limited spacetime covariances of the original non-gravitational actions which leads to a theory of matter and gravity that possess a local internal $SL(2,C)$ symmetry and is generally covariant in the sense that the action is invariant under infinitesimal diffeomorphisms generated by vector fields ζ^μ that vanish at the boundary of the action's integration that act on the dynamical fields χ as $\chi \to \chi + \pounds_\zeta \chi$ [E115]. We note that the gauging procedure does not uniquely fix the gravitational action but rather suggests a family of potential actions, each of which must possess a symmetry under both local $SL(2,C)$ transformations and spacetime diffeomorphisms. Indeed, allowing the gravitational action to consist of terms up to quadratic order in $R^{KL}(\omega)$ and $T^I \equiv de^I + \omega^{IJ} \wedge e_J$ is the approach of Poincaré gauge theory which permits a wealth of interesting phenomenology [E79].

Now we return to the parameterized approach as applied to the actions of the standard model of particle physics. In the place of the non-dynamical field $\bar{e}^I_\mu$ we instead have $\partial_\mu X^I$. Due to the fields X^I now being dynamical, the actions are generally covariant as well as possessing an additional invariance under the following global transformations:

$$X^I \to \Lambda^I{}_J X^J + P^I \tag{44}$$

where $\Lambda_{JI} = \Lambda^{-1}_{IJ}$ (where indices have been lowered with η_{IJ}). We consider the case where only the Lorentz transformation $X^I \to \Lambda^I{}_J X^J$ in (36) is promoted to a local invariance. A derivative that transforms homogeneously under local Lorentz transformations can be constructed:

$$D^{(\mathcal{A})}_\mu X^I = \partial_\mu X^I + \mathcal{A}^I{}_{J\mu}X^J \tag{45}$$

i.e. under a transformation represented by $\Lambda^I{}_J(x)$ we have $D^{(\mathcal{A})}_\mu X^I \to \Lambda^I{}_J D^{(\mathcal{A})}_\mu X^J$ if $\mathcal{A}^I{}_{J\mu} \to \Lambda^I{}_K \mathcal{A}^K{}_{L\mu}(\Lambda^{-1})^L{}_J - \partial_\mu \Lambda^I{}_K(\Lambda^{-1})^K{}_J$. Additionally, a covariant derivative $D^{(\mathcal{A})}_\mu$ acting on Weyl spinors can be defined using $\mathcal{A}^I{}_{J\mu}$ so that, for example $D^{(\mathcal{A})}_\mu \chi^{A'}$ transforms homogeneously under local $SO(1,3) \simeq SL(2,C)$ transformations. Matter actions originally possessing the global Poincaré invariance (44) then possesses a local Lorentz invariance under the replacement $\partial_\mu X^I \to D^{(\mathcal{A})}_\mu X^I$ and $\partial_\mu \Psi \to D^{(\mathcal{A})}_\mu \Psi$ for spinor fields Ψ. To complete the picture it is additionally necessary to introduce an action for the gravity itself. A potential action for gravity is:

$$S_g[\mathcal{A}, X] = \frac{1}{2}\int d^4x \varepsilon^{\mu\nu\alpha\beta}c_{IJKL}D^{(\mathcal{A})}_\mu X^I D^{(\mathcal{A})}_\nu X^J R^{KL}{}_{\alpha\beta}(\mathcal{A}) \tag{46}$$

where

$$c_{IJKL} = \alpha(\epsilon_{IJKL} + 2\beta\eta_{I[K}\eta_{L]J}) \tag{47}$$

$$R^{IJ}{}_{\alpha\beta}(\mathcal{A}) = 2\partial_{[\alpha}\mathcal{A}^{IJ}{}_{\beta]} + 2\mathcal{A}^I{}_{K[\alpha}\mathcal{A}^{KJ}{}_{\beta]} \tag{48}$$

Note that this is the same as the action (9) under the identification $X^I \to \phi^I$, $\mathcal{A}^{IJ}_\mu \to \omega^{IJ}_\mu$ and $g_\pm = \frac{\alpha}{2}\left(1 \mp i\beta\right)$. The gauging of a global symmetry of parameterized field theories does not suggest a unique gravitational field, with additional terms higher order in curvature possible. As a Lagrangian counterpart to the Hamiltonian analysis of [H8], we will see now that four dimensional metric structure and gravitational dynamics described by an extension to GR can emerge from a theory whose gravitational fields are $\{\mathcal{A}^{IJ}_\mu, X^I\}$. Coupling to matter fields χ is implemented by the promotion $\partial_\mu X^I \to D^{(\mathcal{A})}_\mu X^I$ and the use of the Lorentz covariant derivative $D^{(\mathcal{A})}_\mu = \partial_\mu + \frac{1}{2}\mathcal{A}_{IJ\mu}J^{IJ}$ acting on spinors, where J^{IJ} are the generators of $SL(2,C)$:

$$S_m = S_m[\chi^{(m)}, D^{(\mathcal{A})}X^I, \mathcal{A}^{IJ}(J)] = \int \mathcal{L}_m \tag{49}$$

where $\chi^{(m)}$ are matter fields. The Lorentz-invariant $X_I X^I$ scalar is permitted to appear in actions but we do not consider coupling of this quantity to matter - indeed, any actions which have a symmetry under 'covariantly constant' translations $X^I \to X^I + P^I$ subject to $D^{(\mathcal{A})}_\mu P^I = 0$ cannot feature such terms. The promotion $\partial_\mu X^I \to D^{(\mathcal{A})}_\mu X^I$ in matter actions suggests that the quantity $g_{\mu\nu} \equiv \eta_{IJ} D^{(\mathcal{A})}_\mu X^I D^{(\mathcal{A})}_\nu X^J$ will indeed play the role of the spacetime metric tensor. To help show the relation of this model to GR in the Lagrangian formalism we introduce the auxiliary field e^I_μ which is to equal $D^{(\mathcal{A})}_\mu X^I$ 'on shell' and replaces instances of $D^{(\mathcal{A})}_\mu X^I$ in S_g and S_m, with this equality implemented via the use of an action which introduces a Lagrange multiplier three-form field λ_I, which written in the language of differential forms is $S_\lambda = \int \lambda_I \wedge (D^{(\mathcal{A})}X^I - e^I)$ so the total action $S = S_g + S_\lambda + S_m$ takes the form:

$$S[e,\mathcal{A},X,\lambda,\chi^{(m)}] = \int \left[c_{IJKL} e^I \wedge e^J \wedge R^{KL} + \lambda_I \wedge \left(D^{(\mathcal{A})}X^I - e^I\right)\right] + S_m[\chi^{(m)}, e, \mathcal{A}] \tag{50}$$

The equations of motion obtained by varying S with respect to e, $\mathcal{A}$, X, and λ are:

$$-2c_{IJKL} e^J \wedge R^{KL} + \frac{\partial \mathcal{L}_m}{\partial e^I} - \lambda_I = 0 \tag{51}$$

$$-D^{(\mathcal{A})}(c_{IJ[MN]} e^I \wedge e^J) + \frac{\partial \mathcal{L}_m}{\partial \mathcal{A}^{MN}(J)} + \lambda_{[M} X_{N]} = 0 \tag{52}$$

$$D^{(\mathcal{A})}\lambda_I = 0 \tag{53}$$

$$D^{(\mathcal{A})}X^I - e^I = 0 \tag{54}$$

where, for example, $\delta_e \mathcal{L}_m = \frac{\partial \mathcal{L}_m}{\partial e^I} \wedge \delta e^I$. We now assume that $X_I X^I \neq 0$ over the region of spacetime of interest so that we may define a projector orthogonal to X^I: $\mathcal{P}^I{}_J = \delta^I{}_J - \frac{1}{X_K X^K} X^I X_J$. We therefore have that:

$$e^I_\mu = \frac{1}{X^2}\mathcal{E}_\mu X^I + E^I_\mu \tag{55}$$

where $E^I_\mu \equiv \mathcal{P}^I{}_J D^{(\mathcal{A})}_\mu X^J$, hence $X_I E^I_\mu = 0$. By the definition $D^{(\mathcal{A})}_\mu X^I = e^I_\mu$ we have $\mathcal{E}_\mu = \frac{1}{2}\partial_\mu X^2$ and hence:

$$g_{\mu\nu} \equiv \eta_{IJ} e^I_\mu e^J_\nu = \eta_{IJ} D^{(\mathcal{A})} X^I D^{(\mathcal{A})} X^J = \frac{1}{4X^2}\partial_\mu X^2 \partial_\nu X^2 + E^I_\mu E_{I\nu} \tag{56}$$

To cover distinct cases, we can define $X_I X^I = \xi \mathcal{X}^2$ where $\xi = -1$ if $X_I X^I$ is timelike and $\xi = 1$ if $X_I X^I$ is spacelike. It's useful to clarify the signature of the tensor $h_{\mu\nu} \equiv E^I_\mu E_{I\nu}$. The quantities $g_{\mu\nu}$ and $h_{\mu\nu}$ are each Lorentz gauge-independent. For the case $\xi = -1$ and X^I is real, we can find a gauge where $X^I \stackrel{*}{=} \sqrt{-X_J X^J}\delta^I_0$ where $\eta_{00} = -1$; therefore, as $X_I E^I = 0$, the signature of $h_{\mu\nu}$ is $(0,+,+,+)$ and it can be considered as a spatial metric orthogonal to the timelike vector $\partial^\mu X^2$. Alternatively, for the case where $\xi = 1$ and X^I is real, we can find a gauge where $X^I \stackrel{*}{=} \sqrt{X_J X^J}\delta^I_1$ where η_{11}, implying that in this case the signature of $h_{\mu\nu}$ is $(0,-,+,+)$ and it can be considered as a timelike metric orthogonal to the spacelike vector $\partial^\mu X^2$. As we have seen, only for the values $\beta = \pm i$ does a General-Relativistic limit of the theory exist [H2]. For concreteness we consider the case $\beta = i$. By taking the anti-self dual part of (52) and assuming that $\mathcal{A}^{IJ}_\mu(J) = \mathcal{A}^{+IJ}_\mu$ i.e. covariant derivatives of spinor fields are to be built using $\mathcal{A}^{+IJ}_\mu$ (which is a consistent choice for coupling to spinor fields [E21]) it follows that $\lambda_I \propto X_I$ and so we may define the field ϱ via $\lambda_I = (\varrho/\mathcal{X})X_I$, following which the equations of motion become:

$$-4\alpha\epsilon_{IJKL}e^J \wedge R^{KL}(\mathcal{A}^+) + \frac{\partial\mathcal{L}_m}{\partial e^I} - \frac{1}{\mathcal{X}}X_I\varrho = 0 \tag{57}$$

$$-2\alpha D^{(\mathcal{A}^+)}(\epsilon_{IJ[KL]}e^I \wedge e^J)^+ + \frac{\partial\mathcal{L}_m}{\partial\mathcal{A}^{+KL}} = 0 \tag{58}$$

$$\partial_\mu\mathcal{X}\partial^\mu\mathcal{X} = \xi \tag{59}$$

$$d\varrho = 0 \tag{60}$$

$$E_I \wedge \varrho = 0 \tag{61}$$

where indices are raised with $g^{\mu\nu}$, taken to be the matrix inverse of (56). Equation (58) in the absence of coupling of the source term due to spinor coupling to the $\mathcal{A}^{+IJ}_\mu$ field[4] implies [E27] that the solution for the self-dual $\mathcal{A}^{+IJ}_\mu$ is given by the self dual part of the Levi-Civita spin connection $\Gamma^{IJ}_\mu(e,\partial e)$ which is defined to be the solution to the equation $de^I + \Gamma^I{}_J \wedge e^J = 0$ [E27]. We can make contact with standard notation by writing the three-form Einstein equation as a tensor equation:

$$4\alpha\epsilon_{IJKL}\varepsilon^{\mu\nu\alpha\beta}e^J_\mu R^{KL}{}_{\nu\alpha}(\mathcal{A}^+) = \frac{2}{3!}\left[\frac{\partial\mathcal{L}_m}{\partial e^I}\right]_{\mu\nu\alpha}\varepsilon^{\mu\nu\alpha\beta} + \frac{2}{3!}X_I\varrho_{\mu\nu\alpha}\varepsilon^{\mu\nu\alpha\beta} \tag{62}$$

We now make the following ansatz for ϱ which satisfies (61):

$$\varrho_{\mu\nu\alpha} = -\frac{1}{2}\xi\sqrt{-g}\varepsilon_{\mu\nu\alpha\beta}\partial^\beta\mathcal{X}\rho \tag{63}$$

Now, multiplying (62) by e^I_ζ and using $X_I e^I_\zeta = \frac{1}{2}\xi\partial_\zeta\mathcal{X}^2$ as well as defining $\alpha \equiv 1/(64\pi G)$ we have:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} = 8\pi G\big(T^{(m)}_{\mu\nu} + \rho\partial_\mu\mathcal{X}\partial_\nu\mathcal{X}\big) \tag{64}$$

where $\bar{R}^{IJ}{}_{\mu\nu}$ is the curvature two-form associated with $\Gamma^{IJ}_\mu(e,\partial e)$, $R^\mu{}_\nu \equiv R^{\sigma\mu}{}_{\sigma\nu}$, and we've defined the stress energy tensor for matter fields:

$$T^{(m)}_{\mu\nu} = \frac{1}{3!2\sqrt{-g}}e^I_{(\nu}g_{\mu)\sigma}\varepsilon^{\alpha\beta\gamma\sigma}\left[\frac{\partial\mathcal{L}_m}{\partial e^I}\right]_{\alpha\beta\gamma} \tag{65}$$

The equation $d\varrho = 0$ becomes:

$$0 = \partial_\mu\left(\sqrt{-g}\rho\partial^\mu\mathcal{X}\right) \tag{66}$$

We see then from (64) and (66) that when $\xi = -1$, Einstein's equations in the presence of an additional dust-like fluid component with density ρ and four-velocity $V_{(\xi=-1)\mu} = \partial_\mu\mathcal{X}$ are recovered. Additionally it follows from (59) that

$$V^\mu_{(\xi=-1)}\bar{\nabla}_\mu V^\nu_{(\xi=-1)} = 0 \tag{67}$$

where $\bar{\nabla}_\mu$ is the covariant derivative according to the Christoffel symbols $\Gamma^\alpha_{\mu\nu}(g,\partial g)$ i.e. $V^\mu_{(\xi=-1)}$ describes timelike geodesic curves in spacetime. Alternatively, for the case $\xi = 1$ ($X_I X^I > 0$), the equations (64)

[4] The inclusion of such sources will modify the solution for $\mathcal{A}^{+IJ}_\mu$ so that a term involving spinor currents will appear when the metric Einstein equations are ultimately recovered.

and (66) still apply but with a different interpretation: the vector $V^{\mu}_{(\xi=1)}$ is spacelike and, satisfying $V^{\mu}_{(\xi=1)}\bar{\nabla}_{\mu}V^{\nu}_{(\xi=1)} = 0$ the fields describe spacelike geodesic curves in spacetime. As such, the source term due to ρ in (64) is more readily interpreted as a 'dark pressure'. We will see in that there exist simple solutions where in some parts of spacetime $X^I X_I < 0$, in others $X^I X_I > 0$ and in others $X^I X_I = 0$ (either by X^I vanishing or being null) - therefore in such cases the projector $\mathcal{P}^{IJ}$ cannot be globally defined. By way of illustration and comparison to GR, we can consider the action of the present model in FRW symmetry, wherein the action can be shown to be, after elimination of certain components of $\mathcal{A}^{IJ}_{\mu}$ using the equations of motion:

$$S[a,T] \stackrel{b}{=} \frac{3}{8\pi G}\int \sqrt{\bar{h}}d^3xdta^3\dot{T}\Big(-\frac{1}{a^2}\frac{\dot{a}^2}{\dot{T}^2}+\frac{k}{a^2}\Big) \tag{68}$$

Note that the action (68) has reduced to an instance of parameterized particle mechanics which is a reformulation of Newtonian mechanics where the Newtonian time T is promoted to a dynamical field [E25]. We can introduce new fields P_T and N such that P_T is a Lagrange multiplier term enforcing the definition of the 'time velocity' $N = \dot{T}$, with the action becoming:

$$S[P_T,T,N,a,\phi] \stackrel{b}{=} \int \sqrt{\bar{h}}d^3xdt\Big[P_T\Big(\dot{T}-N\Big)+a^3N\Big(\frac{3}{8\pi G}\Big(-\frac{1}{N^2}\frac{1}{a^2}\dot{a}^2+\frac{k}{a^2}\Big)\Big)\Big] \tag{69}$$

Varying T and N we have:

$$\frac{3}{8\pi G N^2}\Big(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\Big) = \frac{P_T}{a^3} \tag{70}$$

$$\dot{P}_T = 0 \tag{71}$$

whilst equations of motion obtained by varying a is identical to those in GR. Equations (70) and (71) arise from the field equations (57) and (60) restricted to FRW symmetry. Additionally, P_T itself is the analogue of the integration constant E - the total energy of the system - in parameterized particle mechanics [E25] and it has an observational effect: it would be interpreted as a DM component in the universe.

We now show that the model admits several distinct field configurations that solve the field equations and result in the spacetime metric being that of Minkowski space. Recall that in the case of the non-gravitational parameterized field theory, a Minkowski metric was recovered via

$$\tilde{\eta}_{\mu\nu} = \eta_{IJ}\frac{\partial X^I}{\partial x^{\mu}}\frac{\partial X^J}{\partial x^{\nu}} \tag{72}$$

We will now show that two distinct solutions in the gravitational theory lead to the recovery of Minkowski space. Remarkably, one of these solutions describes a flat spacetime metric with non-zero curvature $R^{IJ}{}_{\alpha\beta}(\mathcal{A})$. To aid visualization, figures showing the profile of the field X^I in Minkowski space are shown in Figure 5.4.3.

Clearly a solution to the vacuum equations of motion (51)-(54) for $\beta = \pm i$ is if the curvature two-form $R^I{}_J = 0$. Then, one can find a gauge where $\mathcal{A}^{IJ}_{\mu} \stackrel{*}{=} 0$ and in this gauge

$$g_{\mu\nu} \stackrel{*}{=} \eta_{IJ}\partial_{\mu}X^I\partial_{\mu}X^J \tag{73}$$

The equations of motion admit solutions where X^I can coordinatize the entire spacetime such that $g_{\mu\nu} = \eta_{\mu\nu}$. Note that here $X^I = 0$ at a single point in spacetime. Such a solution is not unique. Additional X^I related to the original solution by $X^I \to \Lambda^I{}_J X^J + P^I$ - where $\Lambda^I{}_J \in SO(1,3)_C, P^I \in \mathbb{C}^4$ and $\partial_{\mu}\Lambda^I{}_J = \partial_{\mu}P^I = 0$ - are also solutions i.e. there are a family of solutions related by global complexified Poincaré transformations. An alternative possibility is to consider the case of timelike X^I in FRW symmetry and obtain a solution where the scale factor $a(t) = \text{Cst.}$, implying that the metric tensor $g_{\mu\nu}$ takes Minkowski form. Indeed it can readily be seen that from the equations of motion that the connection $\mathcal{A}^{\pm IJ}_{\mu}$ is non-vanishing, with curvature:

$$R^{0i} = -\frac{1}{t^2}dt\wedge dx^i - \frac{i}{t^2}\epsilon^i{}_{jk}dx^j\wedge dx^k \tag{74}$$

$$R^{ij} = \frac{2}{t^2}dx^i\wedge dx^j - \frac{i}{t^2}\epsilon^{ij}{}_k dt\wedge dx^k \tag{75}$$

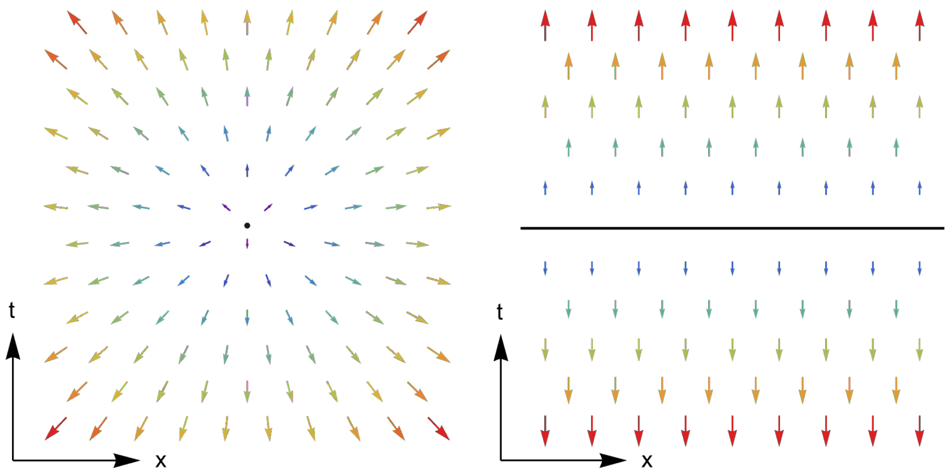

Figure 3: Profiles for X^I in a plane coordinatized by inertial coordinates (x,t) leading to Minkowski metric with zero (left panel) and non-zero (right panel) spacetime curvature respectively. Points where $X^I = 0$ are given in black.

Here X^I now vanishes on the 3-surface $t = 0$. Remarkably, the curvature tensor is non-zero for this solution with flat spacetime metric[5]. Note that the curvature diverges as $t \to 0$. However, nonetheless the gravitational action S_g remains zero for all moments of time as it only depends on the *self-dual* curvature which - as can be verified from (74) and (75) - always vanishes. The existence of solutions with a maximally symmetric metric in the presence of fields which spontaneously break local Lorentz invariance is reminiscent of ghost condensate [E41] and Einstein-Aether [E36] models which permit Minkowski space as a solution to the field equations despite the presence of, respectively, a scalar field ϕ with non-zero time derivative or vector field A^μ with timelike expectation value. The evolution of linear perturbations around these solutions are considered in more detail in [H6, H8].

Finally we note that is also possible to find solutions to the gravitational field equations where the four dimensional metric corresponds to that of flat four dimensional *Euclidean* space i.e. where coordinates exist so that the metric can globally be put in the form $g_{\mu\nu} = \eta_{IJ} D_\mu^{(\mathcal{A})} X^I D_\nu^{(\mathcal{A})} X^J = \mathrm{diag}(1,1,1,1)$. This may be recovered from the non-vanishing curvature Minkowski solution via analytic continuation of $\mathcal{A}_\mu^{\pm IJ}$ or by considering a zero-curvature solution for which in the gauge $\mathcal{A}_\mu^{IJ} \stackrel{*}{=} 0$ we have $X^I = (it, x^i)$ where (t, x^i) comprise a set of inertial coordinates in spacetime. Interestingly the field X^I in the non-zero curvature solution introduces a 'preferred' (imaginary) time coordinate but nonetheless the resultant Euclidean geometry with four dimensional metric $\delta_{\mu\nu}$ possesses symmetry under the group of diffeomorphisms generating global $ISO(4)$ coordinate transformations. In a more general context, consider the case where there exists an $SO(1,3)_C$ gauge where $X^I \equiv iS(x^\mu)\delta_0^I$ where $S(x^\mu)$ is assumed real and hence $X_I X^I = S^2$. In this gauge we have $E_\mu^I \stackrel{*}{=} iSA^I{}_{0\mu}$. Therefore the only non-vanishing E_μ^I are E_μ^i where $i,j,k = 1\ldots 3$ and

$$g_{\mu\nu} = \frac{1}{S^2}\partial_\mu S^2 \partial_\nu S^2 - S^2 \eta_{ij} A^i{}_{0\mu} A^j{}_{0\nu} \tag{76}$$

So if $A^i{}_{0\mu}$ in this gauge are purely imaginary then the spacetime metric is real and of Euclidean signature.

5.4.4 Phenomenology of the model and prospects for further investigation

The extension to GR (30) looks potentially promising: the requirement that a General Relativistic limit only occurs when the self-dual parts of the spatial pullback of the spin connection and its momenta represent

[5]This is the opposite of the case of teleparallel gravity where the spacetime curvature is zero but nonetheless metrics with non-vanishing Riemannian curvature (i.e. curvature built from the Christoffel symbols) exist as solutions to the field equations [E78].

the gravitational degrees of freedom implies that in this limit the gravitational Hamiltonian takes a simple, polynomial form. The presence of the new degree of freedom ϕ^2 in this model produces a gravitational effect equivalent to that of a pressureless perfect fluid in the classical equations of motion. Given that an extremely wide array of cosmological and astrophysical data points towards the presence of an additional, unknown gravitational component that behaves as such a fluid on large scales [E110], it is tempting to speculate whether the new degree of freedom may be responsible for at least some of this effect. A good approximation to all DM models on large, cosmological scales is expected to be the hydrodynamical description in which the DM is described by a fluid with density $\rho(x)$ and four velocity u^μ where the four velocity obeys the geodesic equation according to the metric $g_{\mu\nu}$. Generally a configuration u^μ specified on an initial Cauchy surface will evolve so that $\nabla_\mu u^\mu$ diverges in finite time (the formation of caustics), preventing further evolution of the field via the equation of motion (66) [E71]. It is not difficult to find initial data so that the pathological behaviour arises on timescales orders of magnitude shorter than the age of the universe [E92] and so the viability of model's classical equations of motion is called into question. A possibility is that a *cosmic skeleton* of singular structures would appear in such a scenario; a consequence of this scenario would be that supermassive black holes would form with such ease that the observed mass of the presumed black hole in the centre of the Milky Way galaxies constrains the cosmic abundance of such 'irrotational' DM to be a small fraction of the total amount in our universe [E81].

However, the model (9) should be understood first as a quantum theory. It is possible that if the field ϕ^2 plays a role in determining a privileged and global time in quantum gravity then there may be observable consequences of this. By way of example, one approach [E28, E77] has been to construct the canonical formulation of the action for GR coupled to a pressureless dust component

$$S[g,\rho,\mathcal{X}] = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \rho(\partial_\mu\mathcal{X}\partial^\mu\mathcal{X}+1)\right], \tag{77}$$

and then implement a time gauge fixing constraint $\mathcal{X} \overset{*}{=} t$ prior to quantization. Hence, if $\mathcal{X}$ plays the role of time in the putative quantum theory of gravity (and allowed to flow eternally without obstruction), it is not clear that the caustic pathologies which prevent the use of $\mathcal{X}$ as a global clock in the classical theory can emerge as a limit of the quantum theory. Indeed there is evidence that caustics are indeed avoided when spherical collapse of the pressureless perfect fluid is considered for the model of GR coupled to a pressureless fluid quantized in accordance with [E119]. Additionally, it is known in the context of FRW-symmetric spacetimes, the big bang singularity associated with the classical equations (64) and (66) may be avoided in the quantum theory restricted to this symmetry [E114, E118]. A criticism of such approaches [E26] has been that it has not been clear how degrees of freedom $(\rho, \mathcal{X})$ could appear in a physical theory and we regard it as encouraging that they arise naturally from a theory based on the action (9).

However, it should be emphasized that this approach to the quantization of GR coupled to a pressureless fluid is not universal. Rather, [E58] considered the fluid part of the action decoupled from gravity and constructed the canonical formulation of this part in isolation, recovering a Hamiltonian density $\mathcal{H} = \Pi_{(\mathcal{X})}\sqrt{1+\partial^i\mathcal{X}\partial_i\mathcal{X}}$, where $\Pi_{(\mathcal{X})}$ is the canonical momentum of $\mathcal{X}$ and i denotes a spatial coordinate index which is raised with a flat Euclidean inverse metric. The authors then consider an expansion around a background solution $\mathcal{X} = t$, $\Pi_{(\mathcal{X})} = \rho_0$ $(\partial_\mu\rho_0 = 0)$ with $\delta\mathcal{X} = \chi/\sqrt{\rho_0}$, $\delta\Pi_{(\mathcal{X})} = \Pi_\chi\sqrt{\rho_0}$ where ρ_0 is to be interpreted as the background density of the pressureless perfect fluid. It follows then that $\mathcal{H} = \frac{1}{2}\partial^i\chi\partial_i\chi + \frac{1}{2\sqrt{\rho_0}}\Pi_\chi\partial^i\chi\partial_i\chi + \ldots$, which suggests that the perturbative expansion breaks down for energy scales $\Lambda \sim \rho_0^{1/4}$ which for the current cosmic DM density corresponds to $\Lambda \sim 10^{-3}eV$ suggesting that perturbative quantization of the fluid part of (77) is limited to energy scales $E \ll \Lambda$, which has been argued to be unacceptable for a component of a candidate theory of quantum gravity. It is unlikely that a quantum theory based on this perturbative approach is equivalent to the one based on gauge fixing $\mathcal{X} = t$ prior to quantization.

Another possibility is that new degrees of freedom beyond those present in Model A or (77) become active in regimes close to the formation of caustics, in effect causing the velocity field u_μ to depart from geodesic motion and leading to caustic avoidance. A well-known example of this is the 'UV completion' of (77) in terms of a massive, complex scalar field $\Phi = \lambda e^{i\phi}$. In curved spacetime the Lagrangian for such a field is

$$L_\Phi = \frac{1}{2}\sqrt{-g}(-g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - M^2|\Phi|^2) = \frac{1}{2}\sqrt{-g}\bigg(- \frac{g^{\mu\nu}\partial_\mu\tilde{\lambda}\partial_\nu\tilde{\lambda}}{M^2} - \tilde{\lambda}^2\big(g^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}+1\big)\bigg) \tag{78}$$

where $\tilde{\lambda} = M\lambda$, $\tilde{\phi} = \phi/M$. In the limit $M \to \infty$ and with the identification $\tilde{\lambda}^2 = 2\rho$, $\tilde{\phi} = \mathcal{X}$, we see that L_Φ tends to the form of the fluid part of (77) and indeed it can be shown that solutions for gravity coupled to L_Φ can approach those of (77) for sufficiently large M. For finite M it follows from the $\tilde{\lambda}$ equation of motion that the 'four-velocity' $u_\mu \equiv \partial_\mu\tilde{\phi}/\sqrt{-(\partial_\nu\tilde{\phi}\partial^\nu\tilde{\phi})}$ does not satisfy the geodesic equation and it can be shown that caustics associated with this field do not form [E92]. Thus an alternative to important quantum corrections to the classical equations of motion of Model A or (77) arising would be such a 'UV completion' of the model (46) so as to introduce new degrees of freedom to ameliorate the problem of caustics. Such a scenario is not inconceivable: for example, despite the great success of GR, a leading candidate for cosmic inflation and the origin of structure in the universe is the Starobinsky model of inflation [E12] which considers a correction $\sqrt{-g}R^2$ to the Einstein-Hilbert Lagrangian; this model is equivalent to a scalar tensor theory and the new scalar degree of freedom in gravitation can be of great importance at high energy scales - for example in sourcing large scale structure in the universe [E45].

A final possibility is that the constraint $u_\mu u^\mu + 1 = 0$ with $u_\mu = \partial_\mu\mathcal{X}$ remains in place so that u_μ always satisfies the geodesic equation but that new, additional terms in the action become important close to caustic formation so as to create a repulsive gravity effect, stopping $\nabla_\mu u^\mu$ from diverging. Indeed, a DM effect with a number of similar characteristics to that following from (46) was discovered in the context of the projectable Hořava-Lifshitz gravity [E62, E61] where the four velocity of the DM fluid takes the form $u_\mu = -\partial_\mu T$ where $T(x)$ is a scalar field which acts as a preferred time coordinate in spacetime. It has been argued that caustics should be expected to not form in such theories due to a) corrections to the Lagrangian that depend on the extrinsic curvature of surfaces of constant T (and so may include $\nabla_\mu u^\mu$) which modify classical gravitational dynamics so as to provide a repulsive effect preventing the divergence of $\nabla_\mu u^\mu$ and b) quantum behaviour of the gravitational degrees of freedom, akin to how the big bang singularity may be avoided in minisuperspace quantum cosmological models of a system comprising GR and dust. As we have discussed, behaviour b) may also arise from the action (46) whilst corrections of the type a) are conceivable: it may be checked that equations of motion for the model $\beta = \pm i$ (46) imply that the extrinsic curvature of surfaces of constant $\mathcal{X}$ is contained within the torsion $D^{(\mathcal{A})}e^I = R^I{}_J(\mathcal{A})X^J$ and so additional terms in the action of higher order in these parts of the curvature may be able to dynamically prevent singular behaviour in this extrinsic curvature.

The question of the corrections that should be expected to equations (64) and (66) and how they affect the viability of a DM candidate arising from a description of gravity in terms of a spontaneously-broken gauge theory of the Lorentz group remains an open one. The scenario that geodesic motion is modified by repulsive gravity effects in the vicinity of would-be caustics is perhaps most immediately testable given the effect such a modification would have on the propagation of light, leading to a potential gravitational lensing signature.

We now briefly comment on the coupling of Model A to matter. A surprising result of the model is the existence of solutions with Minkowski spacetime metric yet possessing non-zero gauge field curvature $R^{IJ}(\mathcal{A})$. In a Minkowski coordinate basis (t, x^i), the solution implies that $\mathcal{A}^{+IJ} = 0$, $\mathcal{A}^{-IJ} = (2/t)(n^{[I}E^{J]} + \frac{i}{2}\epsilon^{IJKL}n_K E_L)$ where $n^I = X^I/\sqrt{-X_J X^J}$ where E_L are spatial coordinate basis one-forms satisfying $E_L n^L = 0$ whilst the metric $g_{\mu\nu} = D^{(\mathcal{A})}_\mu X^I D^{(\mathcal{A})}_\nu X_I = \eta_{\mu\nu}$. Any field that couples to $\mathcal{A}^{-IJ}$ in isolation (i.e. aside from the coupling to $\mathcal{A}^{-IJ}$ contained within $g_{\mu\nu}$) will be affected by the background curvature. Here appears an apparent choice in the coupling between spinor fields and gravity. Consider the kinetic term for a $-$ (minus) chirality spinor $\chi^{A'}$. There are two independent possibilities:

$$i\epsilon_{IJKL}e^J \wedge e^K \wedge e^L \wedge \big(\chi^{*A}\sigma^I_{AA'}D^{(\mathcal{A}^-)}\chi^{A'}\big) \tag{79}$$

$$-i\epsilon_{IJKL}e^J \wedge e^K \wedge e^L \wedge \big(D^{(\mathcal{A}^+)}\chi^{*A}\sigma^I_{AA'}\chi^{A'}\big) \tag{80}$$

It is the latter possibility that was considered by Ashtekar et al. [E21] in the self-dual Einstein Cartan theory where the field $\mathcal{A}^{-IJ}$ does not appear in the formalism and hence the term (79) cannot be constructed. It was shown that the coupling (80) nonetheless allowed the recovery of familiar results from the coupling of gravity to spinors in Einstein-Cartan theory. In the present model, if the coupling (80) is chosen then the spinor field does not 'see' the curvature of the background. If, on the other hand, the coupling (79) is chosen then a brief calculation shows that if $\chi^{A'}$ is part of a Dirac spinor Ψ (for example the left handed electron-neutrino in the standard model) then the following couplings appear in the spinor Lagrangian on this background: $a_I\bar{\Psi}\gamma^I\Psi$ and $b_I\bar{\Psi}\gamma^5\gamma^I\Psi$ where $a_I, b_I \sim n_I/t$. Such couplings have been widely studied in the context of Lorentz-violating extensions of the standard model [E33] and contemporary constraints [E113] on the magnitude of components of b_I would correspond to a t value of the order of several months. The fact that $\{a_I, b_I\}$ diverge $t \to 0$ is perhaps indicative that treating the matter coupling to gravity via a term (79) as a small perturbation to the background describing Minkowski space with non-vanishing curvature is not consistent. Nonetheless, it is conceivable that the field configuration $\{X^I, \mathcal{A}^{IJ}\}$ producing the geometry accessible to experiment approximates a part of this solution and so the above Lorentz-violating matter couplings may be relevant, however a definitive answer likely depends on the resolution of the DM propagation issue discussed earlier. A more detailed study of the coupling of Model A to matter with an emphasis on symmetries and conserved Noether currents was recently undertaken [O20].

The model may also have characteristic experimental signatures from early universe physics. For the case of Ashtekar's chiral formulation of gravity (whose canonical formulation arises from (30) in the limit $\tilde{\Pi} \to 0$), it has been argued [E74] that whatever the ultimate form that the quantized theory takes, it should possess a regime which is mappable to a classical cosmological background with metric perturbations describable in terms of the usual inflationary calculation of tensor vacuum quantum fluctuations. Given this assumption, it was found that the primordial spectrum of tensor modes of $+$ and $-$ chirality differed [E74, E68], in contrast to the case of tensor modes in standard inflationary cosmology where gravity is described by metric GR and no such effect exists. Remarkably, such effects in primordial tensor modes may be observable via their effect on the cross-correlation between CMB temperature and polarization fluctuations [E56]. Given this, a first step that could be taken in the case of (30) would be to allow for the effect of additional degrees of freedom $(\tilde{\Pi}, \phi^2)$ in the quantization of cosmological perturbations and see how the above picture is affected.

An alternative approach is to look to describe the behavior of the model as a quantum theory in situations of high symmetry. This has been carried out for Ashtekar's chiral formulation of gravity in the context of loop quantum cosmology [E86, E90, E83] and this approach could be generalized to the case of the model considered here[6].

Finally we briefly discuss possible generalizations of Model A. A natural generalization of (9) would be the introduction of fields (ψ_+, ψ_-) (potentially with non-trivial Lorentz index structure) such that the hitherto constant (g_+, g_-) are reflective of expectation values of these fields. The General-Relativistic limits $(g_+ = 1, g_- = 0)$ and $(g_+ = 0, g_- = 1)$ would then potentially arise from spontaneous symmetry breaking (with the action formally symmetric under the transformations (23) and accompanying transformation of (ψ_+, ψ_-)) and with time variation of the new dynamical fields being of significance in the early universe.

An alternative approach is to look to enlarge the gauge symmetry of the model. It was found in [O18] that a gauge theory of the complexified general linear group $GL(4, C)$ could reduce to (9) following a process of dynamical symmetry breaking.

5.5 Model B

5.5.1 A general class of gravitational theories as alternatives to DM where the speed of gravity always equals the speed of light.

In the absence of direct detection of a particle with the right properties to account for the entirety of DM, it remains a possibility that the effects attributed to DM represent a shortcoming in our understanding of the nature of gravity, GR may not describe gravity accurately in regimes where it was previously expected to - presumably on the large scales where there is evidence for a DM effect. Generally, attempts to extend GR so as to modify the 'force of gravity' on large scales so as to mimic the presence of DM introduce additional fields into the gravitational sector. [E31, E43, E57, E59, E67] introduce additional fields into the gravitational sector whose influence on the visible matter produces DM like effects. Any additional fields

[6] It is interesting to note that despite the classical equivalence of parameterized field theory to the theory of matter fields propagating in fixed Minkowski space background, the Dirac quantization of the former theory faced technical obstacles [E34], the resolution of which required the use of techniques originally developed in the loop quantum gravity paradigm [E55].

coupled non-trivially to the spacetime curvature generally lead to gravitational wave speed different than GR.

Gravitational waves (GW) from the merger of a binary neutron star system have recently been observed by the advanced Laser Interferometer Gravitational Observatory (aLIGO) and the VIRGO interferometer [E91]. Within seconds of this event (GW170817) being detected, a gamma ray burst was independently observed from the same location [E96, E99]. Given the high likelihood that these represent signals from the same event, the specific small time difference - given the large distance between the location of emission (the galaxy NGC 4993) - implies that (in units where the speed of light is unity), the speed of propagation of GW c_T obeys

$$|c_T^2 - 1| \lesssim 10^{-15}. \tag{81}$$

This is a remarkably stringent constraint and has led the exclusion of many modified theories of gravity proposed in order to explain the phenomenon of dark energy [E93, E95, E94, E98, E108]. Equally important is the impact of these observations on gravitational theories functioning as effective DM proxies. This stringent constraint has also been used in [E105] to place constraints on the Einstein-Aether theory [E37].

Early evidence for DM came in the form of observations of the motion of stars within galaxies [E11], where it was found that stars towards the outer regions of galaxies had orbital velocity significantly higher than expected due to the Newtonian gravitational field produced by visible matter. In 1983, Milgrom showed [E15] that this motion of stars could instead result from a modification to the inertia/dynamics of stars at low Newtonian accelerations. Shortly afterwards it was found that these same effects could alternatively result from a non-linear modification to the Poisson equation of Newtonian gravity [E18]. These models are referred to as MOdified Newtonian Dynamics (MOND). To make progress it is clearly necessary to look to recover MOND from a theory that can also account for the experimental successes of GR. Perhaps the most widely-known relativistic theory leading to MOND-like behavior is the Bekenstein-Sanders Tensor-Vector-Scalar (TeVeS) theory [E31, E43] which depends on a metric $\hat{g}_{\mu\nu}$, a unit-timelike (with respect ot the metric $\hat{g}_{\mu\nu}$ vector field A_μ and a scalar field ϕ. All types of matter are taken to couple universally to a metric $g_{\mu\nu}$ via

$$g_{\mu\nu} = e^{-2\phi}\hat{g}_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu \tag{82}$$

and as such the Einstein Equivalence Principle is obeyed. Due to the algebraic relation between the two metrics, there is only one tensor mode propagating gravitational wave perturbation (two polarizations) in this theory just as in GR. The cosmology of TeVeS theory has been extensively investigated in [E49, E51, E88, O17].

The speed of the tensor mode GW in TeVeS theory is in general different than the speed of light, and it is then natural to ask what is the status of the TeVeS paradigm after GW170817. Using a variety of methods, a number of articles [E101, E102, E103] have tackled this question. The authors of [E101] compared the Shapiro time delay of gravitational versus electromagnetic waves, as they pass through the potential wells of galaxies, proposed earlier as a generic test of TeVeS theory [E52]. Such a test is superior to testing the propagation speed on a FRW background spacetime considered in [E93, E95, E94, E98, E105] in the case of other theories. The delay was calculated there by comparing the geodesics of $\hat{g}_{\mu\nu}$ to the geodesics of $g_{\mu\nu}$, however, as the metric is not an observable the generality of their result is unclear. For instance, [O1] reformulated TeVeS theory using a single metric ($g_{\mu\nu}$) so that no geodesic comparisons are possible in that formulation[7]. A different method is necessary In [E102, E103] the speed of all six types of GW present in TeVeS theory [E66] has been considered on a Minkowski background and after imposing (81), analysis of the remaining parameter space led to the conclusion that TeVeS theory is ruled out.

In [H3], the propagation of GW on perturbed FRW spacetimes was investigated, which includes the Shapiro time delay effect. It was shown that the original TeVeS theory [E43] and its generalization [E57, E63] is indeed ruled out by the GW170817/GW170817a events, in agreement with previous studies [E101, E102, E103]. However, it is additionally shown that there exists a previously unknown class of relativistic MOND theories also based on the Tensor-(timelike)Vector-Scalar paradigm, where the speed of gravity always equals the speed of light while retaining the effective bi-metric description leading to the usual MOND phenomenology in galaxies.

[7] Other single-metric theories such as the Horndeski theory studied in [E93, E95, E94, E98, E108] and Einstein-Aether theory studied in [E105] may also yield Shapiro time delay different than the one in GR.

A slight generalization of TeVeS is given by the following action [E57, E63] which depends on the three above fields and the two auxiliary fields λ_A and μ:

$$\hat{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-\hat{g}}\Big[\hat{R} - \hat{K} + \lambda_A\left(A^\rho A_\rho + 1\right) - \mu\hat{g}^{\alpha\beta}\hat{\nabla}_\alpha\phi\hat{\nabla}_\beta\phi - \hat{V}(\mu)\Big] + S_{\rm M}[g] \tag{83}$$

Here G is the bare gravitational constant, $\hat{g}$ and $\hat{R}$ are the determinant and scalar curvature of $\hat{g}_{\mu\nu}$ respectively, $\hat{V}$ is a free function of μ, $S_M[g]$ is the action for all matter fields and $\hat{K} = \hat{K}^{\mu\nu\alpha\beta}\hat{\nabla}_\mu A_\nu\hat{\nabla}_\alpha A_\beta$ is obtained using

$$\hat{K}^{\mu\nu\alpha\beta} = c_1\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta} + c_2\hat{g}^{\mu\nu}\hat{g}^{\alpha\beta} + c_3\hat{g}^{\mu\beta}\hat{g}^{\nu\alpha} + c_4\hat{g}^{\nu\beta}A^\mu A^\alpha \tag{84}$$

The indices of A_μ are always raised using $\hat{g}^{\mu\nu}$, the inverse metric of $\hat{g}_{\mu\nu}$, i.e. $\hat{g}^{\mu\rho}\hat{g}_{\rho\nu} = \delta^\mu{}_\nu$). We emphasise that in contrast to [E57, E63] we allow here the c_I ($I = 1\ldots 4$) to be functions of the scalar field ϕ and this comes out to be very important when analyzing the speed of GW. The original TeVeS theory is obtained when $c_I = \{2K_B - \frac{1}{4}, -\frac{1}{2}, -2K_B + \frac{3}{4}, K_B - \frac{1}{4}\}$, for a constant K_B [E63]. For notational compactness we define $c_{IJ\ldots} \equiv c_I + c_J + \ldots$.

The emergence of MOND behavior in the quasistatic weak field limit in the constant c_I case has been analysed extensively in [E63]. We revisit that analysis here in order to show that it remains unchanged even when c_I are functions of ϕ. In particular, one expands the scalar field as $\phi = \phi_0 + \varphi$ with ϕ_0 a constant and φ time independent. The quasistatic metric is such that $\hat{g}_{00} = -e^{-2\phi_0}(1 - 2\hat{\Psi})$ and $\hat{g}_{ij} = e^{2\phi_0}(1 - 2\hat{\Phi})\gamma_{ij}$. In this coordinate system the vector field has components $A_0 = -e^{-\phi_0}(1 + \hat{\Psi})$ and $A_i = 0$. Using the metric transformation (82) we find the components of the metric $g_{\mu\nu}$ so that

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 - 2\Phi)\gamma_{ij}dx^i dx^j \tag{85}$$

where

$$\hat{\Psi} = \Psi - \varphi, \qquad \hat{\Phi} = \Phi - \varphi, \tag{86}$$

With this ansatz, the vector field equations are identically satisfied while the Einstein and scalar field equations reduce to

$$\vec{\nabla}^2\hat{\Psi} = \frac{8\pi G}{2 - c_1 + c_4}\rho \tag{87}$$

$$\vec{\nabla}_i\big(\mu\vec{\nabla}^i\varphi\big) = 8\pi G\rho \tag{88}$$

$$\hat{\Phi} = \hat{\Psi} \tag{89}$$

where ρ is the matter energy density and where the c_I's are evaluated at $\phi = \phi_0$ in (87). The non-dynamical field μ is obtained via a constraint equation found from the action upon variation wrt μ and this equation depends on the form of $\hat{V}(\mu)$. Not all functions $\hat{V}(\mu)$ lead to either Newtonian or MONDian limiting behaviors and the ones that do so must have appropriate properties discussed in [E63]. In order to determine the speed of propagation of GW, we need the tensor mode equation on an FRW background. We assume a metric $g_{\mu\nu}$ such that

$$ds^2 = -\,dt^2 + a^2\left(\gamma_{ij} + \chi_{ij}\right)dx^i dx^j \tag{90}$$

where a is the scale factor, γ_{ij} is the spatial metric of constant curvature κ and χ_{ij} is the tensor mode GW which is traceless $\gamma^{ij}\chi_{ij} = 0$ and transverse $\vec{\nabla}_i\chi^i{}_j = 0$, where $\vec{\nabla}_i$ is the spatial covariant derivative compatible with γ_{ij}. As we are interested only in the tensor mode, we let the perturbations of ϕ and A_μ to zero so that $\phi = \bar{\phi}(t)$ and $A_0 = -e^{-\bar{\phi}}$ with $A_i = 0$. The perturbed Einstein equations for the tensor mode have been obtained for constant c_I in [E57]. In the case where $c_I = c_I(\phi)$, we find an additional term present such that

$$e^{2\bar{\phi}}\left(1 - c_{13}\right)\left[\ddot{\chi}^i{}_j + \left(3H + 4\dot{\bar{\phi}}\right)\dot{\chi}^i{}_j\right] - e^{2\bar{\phi}}\frac{dc_{13}}{d\phi}\dot{\bar{\phi}}\dot{\chi}^i{}_j - \frac{1}{a^2}e^{-2\bar{\phi}}\left(\vec{\nabla}^2 - 2\kappa\right)\chi^i{}_j = 16\pi G e^{-2\bar{\phi}}\Sigma^{(g)i}{}_j \tag{91}$$

where $\Sigma^{(g)i}{}_j$ is a traceless source term due to matter. The only difference from the constant c_I case is the appearance of the $\frac{dc_{13}}{d\phi}$ term multiplying $\dot{\chi}^i{}_j$. Now in the original and in the generalized TeVeS theories it is clear that the speed of propagation of the tensor mode is given by

$$c_T^2 = \frac{e^{-4\bar{\phi}}}{1 - c_{13}} \tag{92}$$

Thus, in general c_T^2 will differ from unity putting this theory in conflict with the observations that require $c_T^2 \approx 1$ unless some mechanism sets $\bar{\phi}$ to be an approximately constant value at very low redshift and equal to $\bar{\phi} = -\frac{1}{4}\ln(1-c_{13})$. This is highly unlikely but even if possible, we show below that the Shapiro time delay rules this case out. If c_I are functions of ϕ, however, there seems to be enough freedom to change this fact. In particular, the unique choice of

$$c_{13}(\phi) = 1 - e^{-4\phi} \tag{93}$$

transforms (91) into

$$\ddot{\chi}^i{}_j + 3H\dot{\chi}^i{}_j - \frac{1}{a^2}(\vec{\nabla}^2 - 2\kappa)\chi^i{}_j = 16\pi G \Sigma^{(g)i}{}_j \tag{94}$$

which is identical to the tensor mode equation in GR and thus with this choice $c_T^2 = 1$.

We have shown above that the choice (93) leads to GW tensor mode propagation as in GR while maintaining MONDian behavior. When gravitational and electromagnetic wave pass through potential wells generated by matter, however, they incur an additional (Shapiro) time delay and as [E52] proposed, it may be used to put strong constraints on such theories. We thus examine whether the condition (93) is sufficient to ensure tensor mode propagation with $c_T^2 = 1$ even when including the effect of inhomogeneities. In these situations the physical metric $g_{\mu\nu}$ takes the form

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Phi)\left(\gamma_{ij} + \chi_{ij}\right)dx^i dx^j \tag{95}$$

where the hierarchy $\chi_{ij} \ll \Phi, \Psi \sim 10^{-5}$ has been assumed. Furthermore TeVeS's scalar field ϕ takes the form $\phi = \bar{\phi} + \varphi$ (with $\varphi \ll 1$) while the vector field has components $A_0 = -e^{-\bar{\phi}}(1+\hat{\Psi})$ and $A_i = -ae^{\bar{\phi}}\vec{\nabla}_i\alpha$. The expressions (86) relate the potentials between the two frames. Given (95) the metric $\hat{g}_{\mu\nu}$ will not be in diagonal form but will contain terms coming from the vector perturbation α. In general the potentials are assumed to be space and time dependent.

Defining $\mathcal{T}^{ik}{}_j = \vec{\nabla}^i\chi^k{}_j + \vec{\nabla}_j\chi^{ki} - \frac{2}{3}\vec{\nabla}^l\chi^k{}_l\delta^i{}_j$, after a lengthy and tedious calculation, the tensor mode equation for χ_{ij} is found to be

$$\begin{aligned}
&e^{2\bar{\phi}}\left[(1-c_{13})\left(1-2\hat{\Psi}\right) - \frac{dc_{13}}{d\phi}\varphi\right]\ddot{\chi}^i{}_j + e^{2\bar{\phi}}\mathcal{A}\dot{\chi}^i{}_j - \left\{\frac{1}{a^2}e^{-2\bar{\phi}}\left[(1+2\hat{\Phi})\left(\vec{\nabla}^2 - 2\kappa\right) + \vec{\nabla}_k(\hat{\Psi}-\hat{\Phi})\vec{\nabla}^k\right] - \frac{1}{a}\mathcal{B}_k\vec{\nabla}^k\right\}\chi^i{}_j \\
&+ \frac{1}{a^2}e^{-2\bar{\phi}}(1+2\hat{\Phi})\left(\vec{\nabla}^i\vec{\nabla}_k\chi^k{}_j + \vec{\nabla}_j\vec{\nabla}^k\chi^i{}_k - \frac{2}{3}\vec{\nabla}^l\vec{\nabla}_k\chi^k{}_l\delta^i{}_j\right) + \frac{1}{a^2}e^{-2\bar{\phi}}\Big\{\vec{\nabla}_k(\hat{\Psi}-\hat{\Phi})\mathcal{T}^{ik}{}_j \\
&- 2\left[\vec{\nabla}_k\vec{\nabla}_j\left(\hat{\Phi}-\hat{\Psi}\right)\chi^{ik} - \frac{1}{3}\vec{\nabla}_k\vec{\nabla}_l\left(\hat{\Phi}-\hat{\Psi}\right)\chi^{lk}\delta^i{}_j\right]\Big\} + \frac{1}{a}\mathcal{C}^i{}_j = 16\pi G e^{-2\bar{\phi}}(1-2\varphi)\Sigma^{(g)i}{}_j
\end{aligned} \tag{96}$$

where explicit forms of the terms $\mathcal{A}$, $\mathcal{B}_i$, $\mathcal{C}^i{}_j$ can be found in [H3]. Allowing all potentials as well as φ and α to vanish reduces (96) to (91). Consider first the reduction of (96) to quasistatic backgrounds, also ignoring the source term, for the fine-tuned case where $c_{13} = 1 - e^{-4\phi_0}$ (so that $c_T^2 = 1$ on the background). We obtain this by setting $a = 1$, $\bar{\phi} = \phi_0$ and $\hat{\Psi} = \hat{\Phi}$ from (89). Then it can be verified that the functions $\mathcal{A}$, $\mathcal{B}_k$, $\mathcal{C}^i{}_j$ all vanish. In addition, considering LIGO wavelengths $\sim 1000km$ which are far smaller than the scale of the potential wells, we may drop the terms containing derivatives on Ψ and φ, i.e $\partial\Phi \ll \partial\chi$, even with $\chi \ll \Phi$ [E108]. Finally, imposing further the gauge condition $\vec{\nabla}_i\chi^i{}_j = 0$, (96) leads to

$$\left(1-2\hat{\Phi}\right)\ddot{\chi}^i{}_j - \left(1+2\hat{\Phi}\right)\vec{\nabla}^2\chi^i{}_j = 0 \tag{97}$$

Thus in this case we expect a Shapiro time delay dictated by $\hat{\Phi}$, the potential formed by baryons alone. This is not the same as Φ which is the potential seen by photons, hence, this fine-tuned case is ruled out by the analysis of [E101]. Let us turn now to the case where (93) holds so that $c_T^2 = 1$ on FRW backgrounds. Imposing (93) we find $\mathcal{B}_k = 0$ and $\mathcal{C}^i{}_j = 0$. Further using (86) we find $e^{2\bar{\phi}}\mathcal{A} = e^{-2\bar{\phi}}\left[3H(1-2\Psi-2\varphi) - \dot{\Psi} - 3\dot{\Phi}\right]$

and after choosing the gauge condition $\vec{\nabla}_i \chi^i{}_j = \vec{\nabla}_i(\Phi - \Psi)\chi^i{}_j$, (96) turns into

$$
\begin{aligned}
&(1-2\Psi)\left[\ddot{\chi}^i{}_j + \left(3H - \dot{\Psi} - 3\dot{\Phi}\right)\dot{\chi}^i{}_j\right] \\
&- \frac{1}{a^2}(1+2\Phi)\Bigg[\left(\vec{\nabla}^2 - 2\kappa\right)\chi^i{}_j + \vec{\nabla}_j\vec{\nabla}^k(\Phi-\Psi)\chi^i{}_k - \vec{\nabla}^i\vec{\nabla}_k(\Phi-\Psi)\chi^k{}_j \\
&- \vec{\nabla}_k(\Phi-\Psi)\vec{\nabla}^k\chi^i{}_j\Bigg] = 16\pi G\Sigma^{(g)i}{}_j
\end{aligned}
\tag{98}
$$

which is the same equation as in GR. Thus, with the choice (93) the tensor mode propagates at the speed of light even when including inhomogeneities and gives the same Shapiro time delay as for photons. To gain further insight as to why this behavior emerges we consider the single-metric (physically equivalent) formulation of TeVeS. As shown in [O1], one introduces a new field $B_\mu = A_\mu$ leading to $B^\mu = e^{-2\phi}A^\mu$ and writes the Lagrange constraint as $g^{\mu\nu}B_\mu B_\nu \equiv B^2 = -e^{-2\phi}$. This makes it possible to solve for ϕ and thus remove both ϕ and $\hat{g}_{\mu\nu}$ from the action. The number of degrees of freedom remain unchanged as now B_μ contains 4 degrees of freedom rather than the 3 of A_μ. The action $S[g, B, \mu]$ of this physically equivalent Vector-Tensor formulation is

$$
S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - K - U] + S_m[g]
\tag{99}
$$

where $U = \hat{V}(\mu)/B^2$ and K is given by

$$
K = (d_1 - d_3)F^{\mu\nu}F_{\mu\nu} + d_{13}M^{\mu\nu}M_{\mu\nu} + d_2 J^2 + d_4 J^\nu J_\nu + \frac{1}{2}d_5 J^\mu \nabla_\mu B^2 + \frac{d_6}{4}(\nabla B^2)^2 + \frac{d_7}{2}QJ + \frac{d_8}{4}Q^2
\tag{100}
$$

and we have also defined $F_{\mu\nu} = 2\nabla_{[\mu}B_{\nu]}$, $M_{\mu\nu} = 2\nabla_{(\mu}B_{\nu)}$, $J = \nabla_\mu B^\mu$, $J_\mu = B^\alpha\nabla_\alpha B_\mu$ and $Q = B^\alpha\nabla_\alpha B^2$. The functions for the generalised TeVeS theory d_I $(I = 1\ldots 8)$ may be found in the appendix of [E63].

Some of the d_I coefficients depend on μ so that MOND behavior may emerge upon choosing appropriate $\hat{V}$. Allowing for a general dependence $d_I(B^2, \mu)$ in (99) represents a slight generalization of (83). Interestingly, the dynamical tendency towards B_μ having a non-vanishing norm in this picture arises from the presence of inverse powers of the norm B^2 in the Lagrangian, rather than via a Lagrangian constraint as in (83). In this formulation, the modification to the speed of propagation of GW is due entirely to the coupling of gravity to the field B_μ through that field's kinetic term. A straightforward way to see this is by considering the case where B^μ is hypersurface orthogonal; in which case we can decompose the metric $g_{\mu\nu}$ as $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$ where $n_\mu \equiv B_\mu/\sqrt{-B^2} = N\nabla_\mu t$ for some global time function t and $h_{\mu\nu}$ $(h_{\mu\nu}n^\nu = 0)$ is the spatial metric on surfaces of constant time. Then

$$
K = -d_{13}B^2\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - d_2 B^2\mathcal{K}^2 + \ldots
\tag{101}
$$

where we have defined the extrinsic curvature tensor $\mathcal{K}_{\mu\nu} \equiv \frac{1}{2}\mathcal{L}_n h_{\mu\nu}$, and $\ldots$ denote terms of linear order or lower in $\mathcal{K}_{\mu\nu}$. As gravitational wave perturbations reside in 'trace-free' small perturbations to $h_{\mu\nu}$, only the first term in (101), schematically of the form $\sim d_{13}B^2\dot{h}^{\mu\nu}\dot{h}_{\mu\nu}$ will affect the speed of gravity. There will be no deviation from GR if

$$
d_{13} = 0 \qquad \Rightarrow \quad d_1 = -d_3.
\tag{102}
$$

The transformation of (83) into (99) gives $d_{13} = \frac{1-c_{13}}{B^6} - \frac{1}{B^2}$ so that $d_{13} = 0$ iff $c_{13} = 1 - B^4 = 1 - e^{-4\phi}$, which is condition (93).

Therefore, in summary there exists a generic class of relativistic theories of MOND based on the Tensor-(timelike)Vector-Scalar paradigm which retain the property that GW in this class propagate as in GR. The original TeVeS theory is not part of this class and therefore not consistent with gravitational wave constraints. Hence, actions of the form (99) are sufficiently general that they encompass both phenomenologically viable and non-viable models. Viable models are those for which $d_3 = -d_1$ so that the $M_{\mu\nu}$ term is absent while all remaining d_I's can in general be functions of both B^2 and μ. However, not all such viable actions lead to MOND behavior but specific functional forms of d_I do so. Indeed, it is possible to sufficiently simplify the viable subset of (99) while retaining a MOND limit and at the same time giving a realistic cosmology. This subset is explored in detail in the following section.

5.5.2 A new relativistic theory for Modified Newtonian Dynamics

This section presents the first relativistic completion of MOND which reproduces galactic and lensing phenomenology similar to the Bekenstein-Sanders Tensor-Vector-Scalar (TeVeS) theory [E31, E43] and, unlike TeVeS, successfully reproduces the key cosmological observables: the anisotropies of the CMB and the spectrum of matter perturbations. Relativistic models of MOND (RelMOND) have always been constructed on phenomenological grounds rather than based on fundamental principles. Quite likely the reason is that the MOND law is empirical, and even the observation that it is scale invariant in the deep-MOND limit [E30, E60] has not yet led to a definitive conclusion as to how this invariance could lead to a MOND gravitational theory. RelMOND models should obey the principle of general covariance and the Einstein equivalence principle. These are, however, do not provide any guidance as to how such theories should look like. Indeed, many theories obeying these have nothing to do with MOND, and many RelMOND theories obeying these same principles are in conflict with observations. Principle-based MOND theories include [E35, E72, E100], however, these are nonrelativistic. Still, the phenomenological approach, that we also follow, can provide valuable guidance toward a more fundamental theory.

What are the necessary phenomenological facts that any successful MOND theory should lead to? It must (i) return to GR (hence, Newtonian gravity) when the gradient of the weak-field limit gravitational potential $\vec{\nabla}\Phi \gg a_0$ in quasistatic situations while (ii) reproducing the MOND law (1) when $\vec{\nabla}\Phi \ll a_0$. It should also (iii) be in harmony with cosmological observations including the cosmic microwave background (CMB) anisotropies and matter power spectrum (MPS), (iv) reproduce the observed gravitational lensing of isolated objects without DM halos, and (v) propagate tensor mode gravitational waves (GWs) at the speed of light.

We consider each requirement in turn. Clearly, (i) means that when $|\vec{\nabla}\Phi| \gg a_0$, the standard Poisson equation $\vec{\nabla}^2\Phi = 4\pi G_{\rm N}\rho$ holds while (ii) means that when $|\vec{\nabla}\Phi| \ll a_0$ the MOND equation (1) holds. While in many cases [E70, E87, O4] the transition between (i) and (ii) depends only on $|\vec{\nabla}\Phi|$, in TeVeS it is facilitated by a scalar d.o.f. φ. We follow the latter and assume that the physics encapsulated by (i) and (ii) fits within the TeVeS framework. A template nonrelativistic action then, is

$$S = \int d^4x \left\{ \frac{1}{8\pi\hat{G}} \left[|\vec{\nabla}\hat{\Phi}|^2 + \mathcal{J}(\mathcal{Y}) \right] + \Phi\rho \right\}, \tag{103}$$

where $\Phi = \hat{\Phi} + \varphi$ is the potential that couples universally to matter, $\hat{G}$ is a constant and $\mathcal{Y} = |\vec{\nabla}\varphi|^2$. The field φ obeys

$$\vec{\nabla} \cdot [(d\mathcal{J}/d\mathcal{Y})\vec{\nabla}\varphi] = 4\pi\hat{G}\rho \tag{104}$$

while $\hat{\Phi}$ obeys the Poisson equation

$$\vec{\nabla}^2\hat{\Phi} = 4\pi\hat{G}\rho \tag{105}$$

Emergence of MOND is then ensured if $\mathcal{J} \to \frac{2\lambda_s}{3(1+\lambda_s)a_0}\mathcal{Y}^{3/2}$ as $\vec{\nabla}\varphi \to 0$. It is in this limit that a_0 appears.

For a point source of mass M, the MOND-to-Newton transition occurs at $r_M \sim \sqrt{(G_{\rm N}M/a_0)}$. A MOND force $\sim \sqrt{G_{\rm N}Ma_0}/r$ lends its way trivially to a Newtonian force $G_{\rm N}M/r^2$ as $r \ll r_M$ but in the inner Solar System this is not sufficient. Corrections to r^{-2} due to φ will compete with the post-Newtonian force $\sim (G_{\rm N}M)^2/r^3$, and these are constrained at Mercury's orbit to less than $\sim 10^{-4}$ [E82, E106]. Suppressing these may happen either through screening or tracking. In the former, φ is screened at large $\vec{\nabla}\varphi$ so that $\Phi \approx \hat{\Phi}$ while in the latter $\varphi \to \hat{\Phi}/\lambda_s$, so that $G_{\rm N} = (1 + 1/\lambda_s)\hat{G}$. We model both with λ_s since screening is equivalent to $\lambda_s \to \infty$. In terms of $\mathcal{J}$, tracking happens if $\mathcal{J} \to \lambda_s\mathcal{Y}$, while screening occurs if $\mathcal{J}$ has terms $\mathcal{Y}^p$ with $p \geq 3/2$ (this may be in conflict with Mercury's orbit even as $p \to \infty$) or via higher-derivative terms absent from (103).

Consider requirement (iii), that is, successful cosmology. In (103) we have a new d.o.f. $\varphi(\vec{x})$ and we expect that the same will appear in cosmology, albeit with a time dependence, i.e. $\bar{\phi}(t)$. Consider a flat FRW metric so that $g_{00} = -N^2$ and $g_{ij} = a^2\gamma_{ij}$ where $N(t)$ is the lapse function and $a(t)$ the scale factor. What should the expectation for a cosmological evolution of $\bar{\phi}(t)$ be? The MOND law for galaxies is silent regarding this matter. There is, however, another empirical law which concerns cosmology: the existence of sizable amounts of energy density scaling precisely as a^{-3}. Within the DM paradigm such a law is a natural consequence of particles obeying the collisionless Boltzmann equation. The validity of this law has been tested [E104, E116] and during the time between radiation-matter equality and recombination it is valid within an accuracy of $\sim 10^{-3}$. Do scalar field models leading to energy density scaling as $\bar{\rho} \sim a^{-3}$ exist?

The answer is yes: shift symmetric k essence. It has been shown [E44] that a scalar field with Lagrangian $\sim \mathcal{K}(\bar{\mathcal{X}})$ where $\bar{\mathcal{X}} = \dot{\bar{\phi}}^2/N^2$, leads to dust (i.e. $\bar{\rho} \sim a^{-3}$) plus cosmological constant (CC) solutions provided $\mathcal{K}(\bar{\mathcal{X}})$ has a minimum at $\bar{\mathcal{X}} = \mathcal{X}_0 \neq 0$. Such a model is the low energy limit of ghost condensation [E42, E50] although the latter also contains higher derivative terms $\sim (\Box\phi)^2$ in its action. The FRW action is

$$S = \frac{1}{8\pi\tilde{G}} \int d^4x N a^3 \left[-\frac{3H^2}{N^2} + \mathcal{K}(\bar{\mathcal{Q}}) \right] + S_m[g] \tag{106}$$

where $\bar{\mathcal{Q}} = \dot{\bar{\phi}}/N$ and $H = \dot{a}/a$. Interestingly, (103) and (106) are shift symmetric in φ and $\bar{\phi}$ respectively. We propose that a MOND analog on FRW is given by (106) with

$$\mathcal{K} = -2\Lambda + \mathcal{K}_2(\bar{\mathcal{Q}} - \mathcal{Q}_0)^2 + \ldots \tag{107}$$

where Λ is the CC, $\mathcal{K}_2$ and $\mathcal{Q}_0$ parameters and $(\ldots)$ denote higher powers in this expansion. Expanding in $\mathcal{Q} - \mathcal{Q}_0$ rather than $\mathcal{X} - \mathcal{X}_0$ is the most general expansion leading to dust solutions and includes the $\mathcal{K}(\bar{\mathcal{X}})$ case. The CC in this model remains a freely specifiable parameter, just as in the Λ-cold DM (ΛCDM) model. Following [E42, E50], we call this the (gravitational) Higgs phase.

Requirement (iv), that is, correct gravitational lensing without DM, requires a relativistic theory. A minimal theory for RelMOND is a scalar-tensor theory[E18] with the scalar providing for a conformal factor between two metrics. However, since null geodesics are unaltered by conformal transformations, such theories cannot produce enough lensing from baryons in the MOND regime. Sanders solved the lensing problem by changing the conformal into a disformal transformation [E31] using a unit-timelike vector field, incorporated by Bekenstein [E43] into TeVeS. The unit-timelike vector has component $A^0 \sim \sqrt{-g^{00}}$ and this ensures that the two metric potentials are equal (as in GR), so that solutions which mimic DM also produce the correct light deflection.

Meanwhile the anisotropic scaling of the MOND law $\sim |\vec{\nabla}\varphi|^3$ compared with a well-behaved cosmology implying terms like $\dot{\bar{\phi}}^2$ and $\dot{\bar{\phi}}^4$, heuristically implies (gravitational) Lorentz violation. A good way of introducing such an ingredient is via a unit-timelike vector field A_μ, much like the spirit of the Einstein-Æther theory [E7, E38], and TeVeS [E31, E43].

The advanced Laser Interferometer Gravitational Observatory (LIGO) and Virgo interferometers [E91] observed GWs from a binary neutron star merger. Combined with electromagnetic observations [E96, E99], this strongly constrains the GW tensor mode speed to be effectively equal to that of light. By analyzing the tensor mode speed, TeVeS has been shown [E101, E102, E103, H3] to be incompatible with the LIGO-Virgo observations for any choice of parameters. The necessary d.o.f. ϕ and A_μ are also ingredients of TeVeS, only there, a second metric was introduced as a combination of $g_{\mu\nu}$, ϕ and A_μ. In [O1], ϕ and A_μ were combined into a timelike (but not unit) vector B_μ, and it was shown that TeVeS may be equivalently formulated with a single metric $g_{\mu\nu}$ minimally coupled to matter, and B_μ with a noncanonical and rather complicated kinetic term. A general class of theories based on the pair $\{g_{\mu\nu}, B_\mu\}$ was uncovered [H3] where the tensor mode speed equals the speed of light in all situations, satisfying requirement (v).

A subset of the general class of actions considered in [H3] depends on a scalar ϕ and unit-timelike vector A^μ such that

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi\tilde{G}} \left[R - \frac{K_{\mathrm{B}}}{2} F^{\mu\nu}F_{\mu\nu} + 2(2 - K_{\mathrm{B}}) J^\mu \nabla_\mu \phi - (2 - K_{\mathrm{B}})\mathcal{Y} - \mathcal{F}(\mathcal{Y}, \mathcal{Q}) - \lambda(A^\mu A_\mu + 1) \right] + S_m[g] \tag{108}$$

where $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$, $J_\mu = A^\alpha \nabla_\alpha A_\mu$, and the Lagrange multiplier λ imposes the unit-timelike constraint on A_μ. In addition $\mathcal{F}(\mathcal{Y}, \mathcal{Q})$ is a free function of $\mathcal{Q} = A^\mu \nabla_\mu \phi$ and $\mathcal{Y} = q^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ where $q_{\mu\nu} = g_{\mu\nu} + A_\mu A_\nu$ is the three-metric orthogonal to A^μ. Notice that (108) is shift symmetric under $\phi \to \phi + \phi_0$.

On FRW $\phi = \bar{\phi}(t)$ while $A_0 = -N$ and $A_i = 0$, hence $\mathcal{Y} = 0$ and $\mathcal{Q} = \bar{\mathcal{Q}}$. We define $\mathcal{K}(\bar{\mathcal{Q}}) = -\frac{1}{2}\mathcal{F}(0, \bar{\mathcal{Q}})$ so that (108) turns precisely into (106), which we have argued that it satisfies requirement (iii). In the weak-field quasistatic limit, we set $g_{00} = -1 - 2\Psi$ and $g_{ij} = (1 - 2\Phi)\gamma_{ij}$ and assume that A^μ aligns with the time direction so that $A^0 = 1 - \Psi$ and $A^i = 0$. The scalar is expanded as $\phi = \bar{\phi} + \varphi$ with $\dot{\varphi} \ll |\vec{\nabla}\varphi|$ and $\dot{\bar{\phi}}$ may be set to its (late Universe) FRW minimum $\mathcal{Q}_0$. Hence, $\mathcal{Q} = (1 - \Psi)\mathcal{Q}_0$. Then (108) leads to $\Psi = \Phi$ which can be substituted into the action to get

$$S = -\int d^4x \left\{ \frac{2 - K_{\rm B}}{16\pi\tilde{G}} \left[|\vec{\nabla}\Phi|^2 - 2\vec{\nabla}\Phi \cdot \vec{\nabla}\varphi + |\vec{\nabla}\varphi|^2 - \mu^2\Phi^2 + \mathcal{J}(\mathcal{Y}) \right] + \Phi\rho \right\} \tag{109}$$

where $\mathcal{J}(\mathcal{Y}) = \mathcal{F}(\mathcal{Y}, \mathcal{Q}_0)/(2 - K_{\rm B})$. Compared with (103) a new term appears which looks like a "mass term" for Φ, with $\mu = \sqrt{\frac{2\mathcal{K}_2}{2-K_{\rm B}}}\mathcal{Q}_0$. The solution for Φ will be as obtained from (103) only for $r \lesssim r_C$ where $r_C \sim \left(r_M \mu^{-2}\right)^{1/3}$, and oscillatory for $r \gtrsim r_C$. We require $\mu^{-1} \gtrsim 1$Mpc so that MOND behavior according to (103) may still be attained in galaxies. Thus, the quasistatic limit has at least three parameters: λ_s, a_0 and μ.

While matter couples only to Φ, gravity comes with two potentials Φ and φ whose action is not diagonal but contains the mixing term $J^\mu \nabla_\mu \phi \to \vec{\nabla}\Psi \cdot \vec{\nabla}\varphi$. Without the latter, φ decouples and no modification of gravity arises in this situation, apart from $\mu^2\Phi$ which is akin to ghost condensation [E42, E50]. Diagonalizing by setting $\Phi = \hat{\Phi} + \varphi$ and identifying $\tilde{G} = (1 - \frac{K_{\rm B}}{2})\hat{G}$ turns (109) into (103) (plus the $\mu^2\Phi^2$ term). Since, $\Psi = \Phi$, (109) leads to the right lensing whenever the solution for Φ mimics DM. This satisfies requirements (i), (ii) and (iv).

The theory just presented was constructed to lead to a FRW universe resembling ΛCDM. Given a general $\mathcal{K}(\mathcal{Q})$, we define the energy density as $8\pi\tilde{G}\bar{\rho} = \mathcal{Q}\frac{d\mathcal{K}}{d\mathcal{Q}} - \mathcal{K}$ and pressure as $8\pi\tilde{G}\bar{P} = \mathcal{K}$ so that the usual FRW equations are satisfied. The field equation for $\bar{\phi}$ may be integrated once to give $\frac{d\mathcal{K}}{d\mathcal{Q}} = \frac{I_0}{a^3}$ for initial condition I_0. When $\mathcal{K}$ obeys the expansion (107), then $\mathcal{Q} = \mathcal{Q}_0 + I_0/a^3 + \ldots$, so that $\bar{\rho} = \bar{\rho}_0/a^3 + \ldots$, where $8\pi\tilde{G}\bar{\rho}_0 = \mathcal{Q}_0 I_0$. The pressure is $\bar{P} = w_0\bar{\rho}_0/a^6 + \ldots$ where $w_0 = \frac{8\pi\tilde{G}\bar{\rho}_0}{4\mathcal{Q}_0^2\mathcal{K}_2}$ is the equation of state at $a = 1$, that is, $w = w_0/a^3 + \ldots$ so that $\bar{P} = w\bar{\rho}$. A time-varying w implies an adiabatic sound speed $c_{\rm ad}^2 = d\bar{P}/d\bar{\rho} = \frac{d\mathcal{K}/d\mathcal{Q}}{\mathcal{Q}d^2\mathcal{K}/d\mathcal{Q}^2}$ and if $\mathcal{K}$ obeys (107) then $c_{\rm ad}^2 = 2w_0/a^3 + \ldots$. Clearly, $w \geq 0$ and $c_{\rm ad}^2 \geq 0$, where the zero point is reached as $a \to \infty$. As the solution depends on the initial condition I_0, the density $\bar{\rho}$ is not (classically) predicted.

For a proper cosmological matter era in the Higgs phase we need w_0 to be sufficiently small. Observations [E104, E116] give $w \lesssim 0.02$ at $a \sim 10^{-4}$, hence, $w_0 \lesssim 2 \times 10^{-14}$. Meanwhile, $\mu^{-1} \gtrsim$ Mpc in order not to spoil the MOND behavior, leading to $w_0 > \frac{3H_0^2{\rm Mpc}^2\Omega_0}{2(2-K_{\rm B})} \gtrsim 10^{-8}$. Unless the effect of the μ term in (109) is alleviated in some future theory, the Higgs phase cannot be extended too long in the past, and higher terms in (107) must be taken into consideration. Within the present setup, one can arrange this with a function $\mathcal{K}(\mathcal{Q})$ which suppresses w and $c_{\rm ad}^2$ during most of the cosmic evolution. Examples are $\mathcal{K} = 2\mathcal{K}_2\mathcal{Z}_0^2\left[\cosh(\mathcal{Z}) - 1\right]$ ("Cosh function") and $\mathcal{K} = 2\mathcal{K}_2\mathcal{Z}_0^2\left[e^{\mathcal{Z}^2} - 1\right]$ ("Exp function") where $\mathcal{Z} = (\mathcal{Q} - \mathcal{Q}_0)/\mathcal{Z}_0$.

The tight coupling of baryons to photons in the early Universe leads to Silk damping and wipes out all small-scale structure in baryons, preventing the formation of galaxies in the late Universe. Within GR, cold DM sustains the gravitational potentials during the tight coupling period, driving the formation of galaxies and affecting the relative peak heights of the CMB as further corroborated by e.g. the Planck satellite [E111]. Checking whether this theory fits the CMB and MPS spectra requires studying linear fluctuations on FRW.

We consider scalar modes in the Newtonian gauge so that $g_{00} = -(1+2\Psi)$, $g_{0i} = 0$ and $g_{ij} = a^2(1-2\Phi)\gamma_{ij}$ and perturb the scalar as $\phi = \bar{\phi} + \varphi$ and the vector as $A_\mu = \{-1 - \Psi, \vec{\nabla}_i\alpha\}$. The perturbed Einstein, vector and scalar equations, then depend on the new scalar modes φ and α and their derivatives. The shear equation remains as in GR, as do the usual perturbed Boltzmann equations for baryon, photons and neutrinos, since they couple only to $g_{\mu\nu}$.

Setting $\chi \equiv \varphi + \dot{\bar{\phi}}\alpha$, $\gamma \equiv \dot{\varphi} - \dot{\bar{\phi}}\Psi$, $E \equiv \dot{\alpha} + \Psi$ and defining the density contrast δ and momentum divergence θ via

$$\delta \equiv \frac{1 + w}{\dot{\bar{\phi}} c_{\rm ad}^2}\gamma + \frac{1}{8\pi\tilde{G}a^2\bar{\rho}}\vec{\nabla}^2\left[K_{\rm B}E + (2 - K_{\rm B})\chi\right] \tag{110}$$

$$\theta \equiv \frac{\varphi}{\dot{\bar{\phi}}} \tag{111}$$

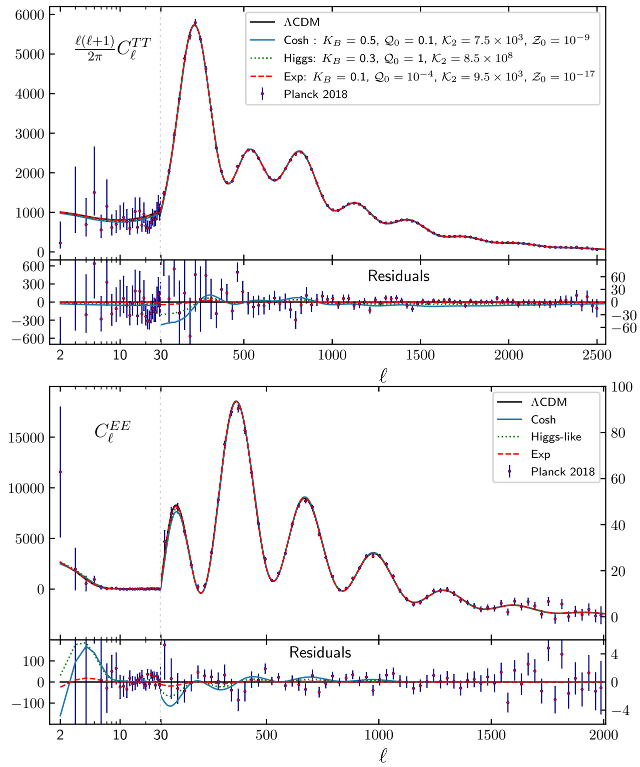

Figure 4: The CMB temperature (T) C_ℓ^{TT} and E-mode polarization C_ℓ^{EE} angular power spectra for ΛCDM and this theory for a collection of functions and parameter values. The ΛCDM parameters are angular acoustic scale $100\theta_s = 1.04171$, DM density $\Omega_c h^2 = 0.1202$, baryon density $\Omega_b h^2 = 0.02235$, reionization optical depth $\tau = 0.049$, helium fraction $Y_{\rm He} = 0.242$, primordial scalar amplitude $10^9 A_s = 2.078$ and spectral index $n_s = 0.963$, while the MOND curves deviate from these within $\sim \{0.07, 0.33, 3.98, 14.29, 1.57, 0.58, 2.60\}$ percent. MOND models have $\lambda_s = \infty$ and their other parameters are shown in the C_ℓ^{TT} panel, with $\mathcal{Q}_0$ and $\mathcal{Z}_0$ in Mpc^{-1}. The "Higgs-like" function parameters are incompatible with a MOND limit.

the Einstein equations take the same form as in GR, i.e. $\delta G^0{}_0 = 8\pi G \sum_I \bar\rho_I \delta_I$ and $\delta G^0{}_j = -8\pi G \sum_I (\bar\rho_I + \bar P_I)\vec\nabla_j \theta_I$ where the index I runs over all matter species including the new variables δ and θ. These obey standard fluid equations

$$\dot\delta = 3H\left(w\delta - \Pi\right) + (1+w)\left(3\dot\Phi - \frac{k^2}{a^2}\theta\right) \tag{112}$$

$$\dot\theta = 3c_{\rm ad}^2 H\theta + \frac{\Pi}{1+w} + \Psi \tag{113}$$

but with nonstandard pressure contrast:

$$\Pi = c_{\rm ad}^2\delta - \frac{c_{\rm ad}^2}{8\pi\tilde G a^2\bar\rho}\vec\nabla^2\left[K_{\rm B}E + (2-K_{\rm B})\chi\right] \tag{114}$$

Hence, the resulting system is not equivalent to a dark fluid: the nonstandard pressure, thus defined, does not close under the fluid variables but, rather, depends on the vector field perturbations α and E. The latter evolves with

$$K_{\rm B}\left(\dot E + HE\right) = \frac{d\mathcal{K}}{d\mathcal{Q}}\chi - (2-K_{\rm B})\left[\frac{\dot{\bar\phi}}{1+w}\Pi + \left(H + \dot{\bar\phi}\right)\chi - 3c_{\rm ad}^2 H\dot{\bar\phi}\alpha\right] \tag{115}$$

Cosmologically, the necessary additional free parameters to ΛCDM are λ_s (influencing the effective cosmological gravitational strength), $K_{\rm B}$, $\mathcal{K}_2$ (or equivalently w_0) and $\mathcal{Q}_0$. These fix μ appearing in the quasistatic regime. More elaborate functions $\mathcal{K}(\mathcal{Q})$ introduce further parameters, e.g. $\mathcal{Z}_0$ in the case of the "Cosh" or

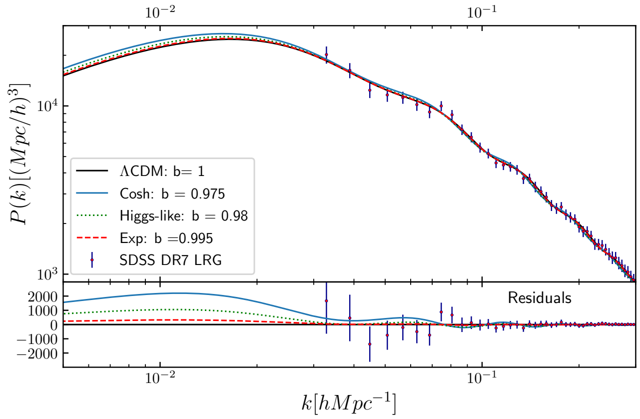

Figure 5: The linear MPS $P(k)$ for the models of Fig. 4 showing excellent fits to the Sloan Digital Sky Survey (SDSS) data release 7 (DR7) luminous red galaxies (LRG) [E65]. We also include a bias parameter b. Note that the (derived) Hubble constant for each model is different.

"Exp" functions above. Note that a_0 does not appear in the linear cosmological regime but will play a role once nonlinear terms from $\mathcal{F}(\mathcal{Y}, \mathcal{Q})$ kick in.

In Figs. 4 and 5 the CMB and MPS are shown in the case of a "Cosh", an "Exp" and a "Higgs-like" function $\mathcal{K}(\mathcal{Q}) = \frac{\mathcal{K}_2}{4\mathcal{Q}_0^2}\left(\mathcal{Q}^2 - \mathcal{Q}_0^2\right)^2$, computed numerically by evolving the FRW background and linearized equations using a) a Boltzmann code written by co-author Constantinos Skordis and b) a version of the CLASS CMB code [E73] modified by myself (Thomas Zlosnik). It has been checked that the results of both codes agree to good accuracy. We have used adiabatic initial conditions and a standard initial power spectrum $P_0 = A_s k^{n_s}$ with amplitude A_s and spectral index n_s. For a wide range of parameters, this relativistic MOND theory is consistent with the CMB measurements from Planck. This happens because $c^2_{\rm ad}$ and w are small enough so that $\Pi \to 0$ and we get dustlike evolution as $\dot{\delta} = 3\dot{\Phi} - \frac{k^2}{a^2}\theta$ and $\dot{\theta} = \Psi$, while the vector field decouples. Note that the field A_μ also contains a 'pure vector'/divergenceless mode perturbation which is expected to behave similarly as in the Einstein-Æther theory [E7, E38]. This may lead to imprints on the B-mode CMB polarization signal.

It has thus been shown that the cosmological regime of this theory reproduces the CMB and MPS power spectra on linear scales and that MOND-like behavior emerges in the quasistatic approximation. The latter is expected to hold for virialized objects, however, how such objects emerge from the underlying density field, i.e. how the two regimes connect, is an open problem. This will happen at a scale which is expected to depend on a_0, μ and $\mathcal{Q}_0$ and quite likely the nonlinear $\sim \nabla(\nabla\phi)^2/a_0$ term coming from $\mathcal{F}$ will play a role. It is reasonable to expect that on mildly nonlinear scales, the quasistatic regime is not yet reached.

Setting $\tilde{M}_p^2 = 1/(8\pi\tilde{G})$ and canonically normalizing as $\tilde{\phi} = \sqrt{2\mathcal{K}_2}\tilde{M}_p\phi$ in (107), the FRW action (106) becomes

$$S = \int d^4x N a^3 \left[-3\tilde{M}_p^2 \frac{H^2}{N^2} + \frac{1}{2}\left(\frac{\dot{\tilde{\phi}}}{N} - \Lambda_c^2\right)^2 + \dots \right] \tag{116}$$

where $\Lambda_c^2 = \tilde{M}_p\sqrt{2\mathcal{K}_2}\mathcal{Q}_0$. Considering the MOND limit in (108) gives $\tilde{M}_p^2\mathcal{F}/2 \to |\vec{\nabla}\tilde{\phi}|^3/\Lambda_0^2$ where $\Lambda_0^2 = 12\left[\mathcal{K}_2(1+1/\lambda_s)/(2-K_{\rm B})\right]^{3/2} M_p a_0$. This scale is indicative of the energy scale above which quantum corrections may be important and below which we can trust the classical theory. Since $a_0 \sim H_0/6$ then $\Lambda_0 \gtrsim \text{meV} \sim (0.1\text{mm})^{-1}$. Newton's r^{-2} law has been tested down to $\sim 52\mu$m [E117] and the curves in Figs.4 and 5 have $\Lambda_0^{-1} \lesssim 100$nm.

Absence of ghosts to quadratic order signifies a healthy theory that could arise as a limit of a more fundamental theory. We do not have such a theory at present but we discuss a case that may bring us closer. The vector in (108) does not seem to obey gauge invariance but in the quadratic action (128) it does so through mixing with diffeomorphisms of $h_{\mu\nu}$. This is not an accident. Let us normalize via $\hat{A}_\mu = M_{\rm ggc}A_\mu$ for some scale $M_{\rm ggc}$ and insert the term $-\frac{1}{4}\frac{\tilde{M}_p^4}{M_{\rm ggc}^4}\lambda^2$. Varying with λ and using the constraint to eliminate λ from the action, perform a Stückelberg transformation $\hat{A}_\mu \to \hat{A}_\mu + \nabla_\mu\xi/M_{\rm ggc}$ and define the covariant derivative acting on "angular field" ξ as $\mathcal{D}_\mu\xi = \nabla_\mu\xi/M_{\rm ggc} + \hat{A}_\mu$. The action turns to

$$S = S_{EH} + \int d^4x\sqrt{-g}\Big\{ - \frac{1}{4g_{\rm ggc}^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}(\mathcal{D}^\mu\xi\mathcal{D}_\mu\xi + M_{\rm ggc}^2)^2 \Big\} \tag{117}$$

plus ϕ-dependent terms, where $F_{\mu\nu} = \nabla_\mu \hat{A}_\nu - \nabla_\nu \hat{A}_\mu$, $g_{\rm ggc}^2 = \frac{M_{\rm ggc}^2}{K_{\rm B}\tilde{M}_p^2}$. The resulting action is that of the gauged ghost condensate (GGC) [E47] or bumblebee field [E22] which has been proposed as a healthy gauge-invariant theory of spontaneous Lorentz violation. The Einstein-Æther theory, part of (108), is the (healthy) decoupling limit of GGC by taking $M_{\rm ggc} \to \infty$ if $0 < K_{\rm B} < 2$ (in our notation) [E47]. It is argued [E47] that $M_{\rm ggc}$ can be as high as 10^{12}GeV. Given that ϕ is shift symmetric it is natural to charge it under this symmetry similar to ξ letting $\mathcal{D}_\mu\phi = \nabla_\mu\phi/M_{\rm ggc} + \hat{A}_\mu$. Interestingly, we may identify $\mathcal{Q} - \mathcal{Q}_0 \to \mathcal{D}^\mu\xi\mathcal{D}_\mu\phi$ while the term $J^\mu\nabla_\mu\phi \to F^{\mu\nu}\mathcal{D}_\mu\xi\mathcal{D}_\nu\phi$, both multiplied by appropriate constants. The terms involving $\mathcal{Y}$ may be constructed using $\left(g^{\mu\nu} + \mathcal{D}^\mu\xi\mathcal{D}^\nu\xi/M_{\rm ggc}^4\right)\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi$. Although extending our work as such does not explain the MOND term $\mathcal{Y}^{3/2}$, it may provide promising directions for further improvements. In the following section, a systematic study of the stability of the model with respect to linear perturbations around Minkowski space is presented.

5.5.3 Aether scalar tensor theory: Linear stability on Minkowski space

As mentioned, typically theories of gravity different to GR (GR) introduce new degrees of freedom into the gravitational sector beyond the metric tensor present in GR [E64, E75]. While these degrees of freedom may have an important role to play in explaining aspects of the DS, it is crucial that they do not also introduce instabilities that are incompatible with observation. Observational constraints suggest that there exist regions of spacetime that can be approximated by highly symmetric solutions (for example geometry in the solar system can be described as a perturbed Minkowski spacetime, whereas the late universe on the largest scales can be described as perturbed de Sitter spacetime) and that these approximations persist for a proper time at least of the order τ_s (for example lower bounds on the age of the solar system or the period of Λ-domination in cosmology).

It is vital then that new degrees of freedom do not introduce instabilities that grow on timescales $\tau_i \ll \tau_s$. To probe this question, one can consider the propagation of small perturbations to the aforementioned highly symmetric solutions. Classically, some theories of gravity allow perturbative modes that grow exponentially, where the timescale τ_i of growth may depend on basic parameters in the theories which can lead to significant constraints on their viability [E54]. Another possibility is that around some backgrounds, some perturbative modes can carry negative energy –either via wrong-sign kinetic terms (ghosts) or wrong-sign mass terms (tachyons). The former especially can signal pathological behavior in the quantum theory of these perturbations, signaling at the least that the background solution cannot be considered stable. If experimental constraints suggest that approximations to the background are long lived then this suggests that the theory of gravity in question is not healthy. Such considerations are therefore vital when considering the viability of a gravitational theory.

The new degrees of freedom in AeST combine with the metric to produce modified Newtonian dynamics (MOND) phenomenology [E15, E18] in the quasistatic, weak-field limit relevant to galaxies while accounting for precision cosmological data [E110] comparably well to the cold DM (CDM) paradigm [8]. The CDM-like cosmological behavior is unrelated to MOND but it is due to terms involving the new fields which have the same form as shift-symmetric k-essence and ghost condensate model Scherrer2004,ArkaniHamedEtAl2003. This results in its cosmological energy density $\propto (1+z)^3$ plus small decaying corrections which makes fitting large scale cosmological data possible.

On a flat FRW background the metric takes the form $ds^2 = -dt^2 + a^2\gamma_{ij}dx^idx^j$ where $a(t)$ is the scale factor and γ_{ij} is a flat spatial metric. The vector field reduces to $A^\mu = (1,0,0,0)$ while $\phi \to \bar{\phi}(t)$ leading to $\mathcal{Q} \to \bar{\mathcal{Q}} = \dot{\bar{\phi}}$ and $\mathcal{Y} \to 0$, so that we may define $\mathcal{K}(\bar{\mathcal{Q}}) \equiv -\frac{1}{2}\mathcal{F}(0,\bar{\mathcal{Q}})$. We require that $\mathcal{K}(\bar{\mathcal{Q}})$ has a minimum at $\mathcal{Q}_0$ (a constant) so that we may expand it as $\mathcal{K} = \mathcal{K}_2\left(\bar{\mathcal{Q}} - \mathcal{Q}_0\right)^2 + \ldots$, where the $(\ldots)$ denote higher terms. This condition leads to $\bar{\phi}$ contributing energy density scaling as dust $\sim a^{-3}$ akin to [E42, E44], plus small corrections which tend to zero when $a \to \infty$. In principle, $\mathcal{K}$ could be offset from zero at the minimum

[8] See [E20, E31, E43, E46, E48, O4, E53, E59, E67, E70, E87, E85, E84, E97, E107, E109, E112] for alternative approaches to the construction of relativistic theories of gravity that contain MOND phenomenology.

$\mathcal{Q}_0$, i.e. $\mathcal{K}(\mathcal{Q}_0) = \mathcal{K}_0$, however, such an offset can always be absorbed into the cosmological constant Λ and thus we choose $\mathcal{K}_0 = 0$ by convention, implying the same on the parent function $\mathcal{F}$.

In the quasistatic weak-field limit we may set the scalar time derivative to be at the minimum $\mathcal{Q}_0$, as is expected to be the case in the late universe. This means that we may expand $\phi = \mathcal{Q}_0 t + \varphi$. Moreover, in this limit $\mathcal{F} \to (2 - K_{\rm B})\mathcal{J}(\mathcal{Y})$, with $\mathcal{J}$ defined appropriately as $\mathcal{J}(\mathcal{Y}) \equiv \frac{1}{2-K_{\rm B}}\mathcal{F}(\mathcal{Y}, \mathcal{Q}_0)$. It turns out that MOND behavior emerges if $\mathcal{J} \to \frac{2\lambda_s}{3(1+\lambda_s)a_0}|\mathcal{Y}|^{3/2}$ where a_0 is Milgrom's constant and λ_s is a constant which is related to the Newtonian/GR limit. Specifically, there are two ways that GR can be restored: (i) screening and (ii) tracking. In the former, the scalar is screened at large gradients $\vec{\nabla}\varphi$, where $\vec{\nabla} \leftrightarrow \vec{\nabla}_i$ is the spatial gradient on a flat background γ_{ij}, and in the latter, $\lambda_s\varphi$ becomes proportional to the Newtonian potential, leading to an effective Newtonian constant

$$G_{\rm N} = \frac{1 + \frac{1}{\lambda_s}}{1 - \frac{K_{\rm B}}{2}}\tilde{G}. \tag{118}$$

Screening may be achieved either through terms in $\mathcal{J} \sim \mathcal{Y}^p$ with $p > 3/2$ or through Galileon-type [E67] terms which must be added to (108). Either way, for our purposes in this section, we may model screening as $\lambda_s \to \infty$.

We are interested in spacetime regions which are well approximated by weak gravitational fields modeled as fluctuations on a Minkowski background $\eta_{\mu\nu}$ and that these regions exist in the late universe where the time derivative of the background field has settled in its minimum $\mathcal{Q}_0$, i.e. $\dot{\bar{\phi}} \to \mathcal{Q}_0$. In addition, the size of these regions is taken to be much smaller than the size of the current cosmological horizon so that we may safely ignore the cosmological constant.

We expand the metric as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $\eta_{00} = -1$ and $\eta_{ij} = \gamma_{ij}$, the vector field [9] as $A_\mu = (-1 + \frac{1}{2}h^{00}, \vec{A}_i)$ and the scalar as $\phi = \mathcal{Q}_0 t + \varphi$. Thus our degrees of freedom are the metric perturbation $h^{\mu\nu}$, vector field perturbation $\vec{A}_i$ (only its 3-dimensional part remains free) and the scalar field perturbation φ, all of which are in general functions of both space and time. We raise/lower spatial indices with the spatial metric γ_{ij}, i.e. $\vec{A}^i = \gamma^{ij}\vec{A}_j$ and set $|\vec{A}|^2 = \vec{A}\cdot\vec{A} = \vec{A}_i\vec{A}^i$ (and use similar notation for other spatial vectors).

Our perturbative variables are amenable to spacetime gauge transformations generated by a vector field ξ^μ. Generally, for a tensor $\mathbf{Y}$, its perturbation $\delta\mathbf{Y} = \mathbf{Y} - \bar{\mathbf{Y}}$ from its background form $\bar{\mathbf{Y}}$ transforms as $\delta\mathbf{Y} \to \delta\mathbf{Y} + \mathcal{L}_\xi\bar{\mathbf{Y}}$, where $\mathcal{L}_\xi$ is the Lie derivative. Usually, on Minkowski space only the metric has a nonzero background value ($\eta_{\mu\nu}$), so that other fields besides the metric perturbation are gauge invariant on such a background; this is typical of dark fields, i.e. additional degrees of freedom which contribute to the energy density but do not mix with the metric perturbation through gauge transformations of this kind. In the present case, however, both the vector field and the scalar field have nonzero background value: $\bar{A}_\mu = (-1, 0, 0, 0)$ and $\bar{\phi} = \mathcal{Q}_0 t$, hence, their perturbations do transform. Specifically, parameterizing ξ^μ as $\xi^\mu = (\xi_T, \vec{\xi}^i)$, we have the usual metric gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu \tag{119}$$

where $\bar{\nabla}_\mu$ is the covariant derivative associated with the Minkowski metric $\eta_{\mu\nu}$. In $3+1$ form the above transformations are explicitly given as

$$h_{00} \to h_{00} - 2\dot{\xi}_T \tag{120}$$

$$h_{0i} \to h_{0i} + \dot{\vec{\xi}}_i - \vec{\nabla}_i\xi_T \tag{121}$$

$$h_{ij} \to h_{ij} + \vec{\nabla}_i\vec{\xi}_j + \vec{\nabla}_j\vec{\xi}_i. \tag{122}$$

The perturbations $\vec{A}$ and φ transform as

$$\vec{A} \to \vec{A} - \vec{\nabla}\xi_T \tag{123}$$

$$\varphi \to \varphi + \mathcal{Q}_0\xi_T. \tag{124}$$

[9] Strictly speaking, to satisfy the Lagrange constraint we need A_0 to second order, i.e. $A_0 = -1 + \frac{1}{2}h^{00} - \frac{3}{8}(h^{00})^2 - \frac{1}{2}|\vec{A}|^2 - h^{0i}\vec{A}_i$ and similarly for A^0. However, for all the other terms in (108), it is sufficient to expand A_0 and A^0 to first order.

Notice how the vector field transformation has the same form as gauge transformations in electromagnetism, however, the generator here is also a diffeomorphism. With these gauge transformations at hand we can create the following gauge-invariant variables:

$$\{\vec{\nabla}\varphi + \mathcal{Q}_0\vec{A}, \dot{\vec{A}} - \frac{1}{2}\vec{\nabla}h_{00}, \dot{\varphi} + \frac{1}{2}\mathcal{Q}_0 h_{00}\} \tag{125}$$

Hence, the fields φ and $\vec{A}_i$ non-trivially mix with the metric perturbation through ξ_T. Our aim is to then expand the action (108) to second order in these fields. With these considerations, and having in mind the discussion in the previous section, we then expand the function $\mathcal{F}$ as

$$\mathcal{F} = (2 - K_{\rm B})\lambda_s\mathcal{Y} - 2\mathcal{K}_2\left(\mathcal{Q} - \mathcal{Q}_0\right)^2 + \dots \tag{126}$$

since $\bar{\mathcal{F}}(0, \mathcal{Q}_0) = 0$ by convention and $\frac{\partial\bar{\mathcal{F}}}{\partial\mathcal{Q}}\big|_{\{0,\mathcal{Q}_0\}} = 0$ at the minimum. The terms denoted by $(\dots)$ are higher order terms which do not contribute to the second order action. We particularly note that one of these is the MOND-type term $\sim |\mathcal{Y}|^{3/2}$ as discussed in the previous section. This term does not contribute to the second order action but we return to it in the discussion section. As an example, consider the function

$$\mathcal{F} = -\,2\mathcal{K}_2(\mathcal{Q} - \mathcal{Q}_0)^2 + \lambda_s\left\{\mathcal{Y} - 2a_0(1+\lambda_s)\sqrt{\mathcal{Y}} + 2(1+\lambda_s)^2a_0^2\ln\left[1 + \frac{\sqrt{\mathcal{Y}}}{(1+\lambda_s)a_0}\right]\right\} \tag{127}$$

In the large $\mathcal{Y}$ limit, the expansion (178) is recovered and the leading correction is $\sim\sqrt{\mathcal{Y}}$, while in the small $\mathcal{Y}$ limit, the expansion is consistent with (178) upon setting $\lambda_s = 0$ and the leading correction is the MOND term $\frac{2\lambda_s}{3(1+\lambda_s)a_0}|\mathcal{Y}|^{3/2}$ Notice the presence of λ_s as a relic of its influence on the observed value of Newton's constant in strong gravity regimes. Expanding (108) to second order leads to

$$\begin{aligned} S = \int d^4x\Big\{ &-\frac{1}{2}\bar{\nabla}_\mu h\bar{\nabla}_\nu h^{\mu\nu} + \frac{1}{4}\bar{\nabla}_\rho h\bar{\nabla}^\rho h + \frac{1}{2}\bar{\nabla}_\mu h^{\mu\rho}\bar{\nabla}_\nu h^\nu{}_\rho - \frac{1}{4}\bar{\nabla}^\rho h^{\mu\nu}\bar{\nabla}_\rho h_{\mu\nu} + K_{\rm B}|\dot{\vec{A}} - \frac{1}{2}\vec{\nabla}h^{00}|^2 \\ &- 2K_{\rm B}\vec{\nabla}_{[i}A_{j]}\vec{\nabla}^{[i}A^{j]} + (2-K_{\rm B})\left[2(\dot{\vec{A}} - \frac{1}{2}\vec{\nabla}h^{00})\cdot(\vec{\nabla}\varphi + \mathcal{Q}_0\vec{A}) - (1+\lambda_s)|\vec{\nabla}\varphi + \mathcal{Q}_0\vec{A}|^2\right] \\ &+ 2\mathcal{K}_2\left|\dot{\varphi} + \frac{1}{2}\mathcal{Q}_0h^{00}\right|^2 + \frac{1}{\tilde{M}_p^2}T_{\mu\nu}h^{\mu\nu}\Big\} \end{aligned} \tag{128}$$

where for convenience we have rescaled the action $S \to 16\pi\tilde{G}S$. We have also omitted the determinant $\sqrt{\gamma}$ in the measure since we are dealing with integrals on Minkowski spacetime, but can be understood to be present in all integrations. Next it is useful decompose the fields into scalar, vector and tensor modes as

$$h_{00} = -\,2\Psi \tag{129}$$

$$h_{0i} = -\,\vec{\nabla}_i\zeta - W_i \tag{130}$$

$$h_{ij} = -\,2\Phi\gamma_{ij} + D_{ij}\nu + 2\vec{\nabla}_{(i}V_{j)} + H_{ij} \tag{131}$$

$$\vec{A} = \vec{\nabla}\alpha + \vec{\beta} \tag{132}$$

where $D_{ij} = \vec{\nabla}_i\vec{\nabla}_j - \frac{1}{3}\gamma_{ij}\vec{\nabla}^2$ is a traceless spatial derivative operator compatible with the background flat Euclidean metric. The modes $\vec{W}$, $\vec{V}$ and $\vec{\beta}$ are pure vector modes, that is, they are transverse: $\vec{\nabla}\cdot\vec{W} = \vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot\vec{\beta} = 0$, while the mode H_{ij} is a pure tensor mode, that is, transverse and traceless: $\vec{\nabla}_iH^i{}_j = H^i{}_i = 0$. The matter stress-energy tensor $T_{\mu\nu}$ is likewise decomposed as

$$T_{00} = \rho \tag{133}$$

$$T_{0i} = \vec{\nabla}_i\theta + p_i \tag{134}$$

$$T_{ij} = P\gamma_{ij} + D_{ij}\Sigma^{(\rm S)} + 2\vec{\nabla}_{(i}\Sigma^{(\rm V)}_{j)} + \Sigma^{(\rm T)}_{ij} \tag{135}$$

where the scalar modes are the matter density ρ, momentum divergence θ, pressure P and scalar shear $\Sigma^{(\rm S)}$, the vector modes are the matter vorticinal momentum density p_i and vector shear $\Sigma^{(\rm V)}_i$, such that $\vec{\nabla}\cdot\vec{p} = \vec{\nabla}\cdot\vec{\Sigma}^{(\rm V)} = 0$, and the tensor mode is the tensor shear $\Sigma^{(\rm T)}_{ij}$, such that $\vec{\nabla}_i\Sigma^{(\rm T)i}{}_j = \Sigma^{(\rm T)i}{}_i = 0$.

With this decomposition, the second order action splits into three distinct parts: one for the scalar modes $S^{(\mathrm{S})}$, one for the vector modes $S^{(\mathrm{V})}$ and one for the tensor modes $S^{(\mathrm{T})}$. We consider each of these three one by one.

The perturbations to fields A_μ and ϕ do not contribute any tensor mode components and so the tensor mode action takes the form:

$$S^{(\mathrm{T})} = \int d^4x \Big\{ \dot{H}^{ij}\dot{H}_{ij} - \vec{\nabla}^k H^{ij}\vec{\nabla}_k H_{ij} + 32\pi\tilde{G}\Sigma^{(\mathrm{T})}_{ij}H^{ij}\Big\} \tag{136}$$

This corresponds to the action for tensor modes present in GR, a result consistent with the earlier, more general calculation that tensor modes in the superclass of theories of which (108) is a special subset, propagate at the speed of light [H3].

Now consider vector modes, which are described by the action

$$S^{(\mathrm{V})} = \int d^4x \Big\{ -\frac{1}{2}\left(\dot{V}_i + W_i\right)\vec{\nabla}^2\left(\dot{V}^i + W^i\right) + K_B\left[|\dot{\vec{\beta}}|^2 - \vec{\nabla}_i\beta_j\vec{\nabla}^i\beta^j - \mathcal{M}^2|\vec{\beta}|^2\right] + 16\pi\tilde{G}\left(\vec{p}\cdot\vec{W} - \Sigma^{(\mathrm{V})}_i\vec{\nabla}^2V^i\right)\Big\} \tag{137}$$

where

$$\mathcal{M}^2 = \frac{(2-K_{\mathrm{B}})(1+\lambda_s)\mathcal{Q}_0^2}{K_{\mathrm{B}}} \tag{138}$$

The field $\vec{\beta}$ decouples from the metric fields $\vec{V}$ and $\vec{W}$ and describes two massive degrees of freedom with mass $\mathcal{M}$. Clearly then we must require $K_{\mathrm{B}} > 0$ to avoid ghosts and gradient instabilities. The mass term $\mathcal{M}$ is also non-tachyonic if both $0 < K_{\mathrm{B}} < 2$ and $\lambda_s > -1$. Hence, stability considerations for the vector modes imply the following constraints on the parameter space of AeST :

$$0 < K_{\mathrm{B}} < 2, \quad \lambda_s > -1. \tag{139}$$

Notice that to this order, the vector modes $\vec{\beta}$ do not couple to matter and thus they are not expected to be generated by sources to leading order.

Finally consider scalar perturbations. Considering only scalar modes in (128) and after some integrations by parts we find the action $S^{(\mathrm{S})}$:

$$\begin{aligned} S^{(\mathrm{S})} = \int d^4x \Big\{ & 6\left(\frac{1}{6}\vec{\nabla}^2\dot{\nu} - \dot{\Phi}\right)\left(\frac{1}{6}\vec{\nabla}^2\dot{\nu} + \dot{\Phi}\right) + 4\left(\frac{1}{6}\vec{\nabla}^2\dot{\nu} + \dot{\Phi}\right)\vec{\nabla}^2\zeta + 2|\vec{\nabla}\Phi|^2 - \frac{2}{3}\Phi\vec{\nabla}^4\nu + 4\left(\vec{\nabla}^2\Phi + \frac{1}{6}\vec{\nabla}^4\nu\right)\Psi \\ & + \frac{1}{18}|\vec{\nabla}(\vec{\nabla}^2\nu)|^2 + 2\mathcal{K}_2\dot{\varphi}^2 - 4\mathcal{K}_2\mathcal{Q}_0\dot{\varphi}\Psi + 2\mathcal{K}_2\mathcal{Q}_0^2\Psi^2 + K_{\mathrm{B}}|\vec{\nabla}(\dot{\alpha}+\Psi)|^2 + 2\left(2-K_{\mathrm{B}}\right)\vec{\nabla}\left(\dot{\alpha}+\Psi\right)\cdot\vec{\nabla}\chi \\ & - \left(2-K_{\mathrm{B}}\right)\left(1+\lambda_s\right)|\vec{\nabla}\chi|^2 - 16\pi\tilde{G}\rho\Psi - 16\pi\tilde{G}\vec{\nabla}^2\theta\,\zeta - 48\pi\tilde{G}P\Phi + \frac{16\pi\tilde{G}}{3}\vec{\nabla}^4\Sigma\,\nu\Big\} \end{aligned} \tag{140}$$

where the gauge-invariant variable $\chi\varphi + \mathcal{Q}_0\alpha$ has been introduced that will be shown to play a prominent role in what follows. Setting scalar matter sources to vanish and moving to Fourier space we have

$$\begin{aligned} S^{(\mathrm{S})} = \int dt \frac{d^3k}{(2\pi)^3}\Big\{ & -6|\dot{\Phi}|^2 + \frac{1}{6}k^4|\dot{\nu}|^2 + 2k^2\left[\left(\frac{1}{6}k^2\dot{\nu} - \dot{\Phi}\right)\zeta^* + c.c.\right] + 2k^2|\Phi - \frac{1}{6}k^2\nu|^2 + 2\mathcal{K}_2|\dot{\varphi} - \mathcal{Q}_0\Psi|^2 \\ & + K_{\mathrm{B}}k^2|\dot{\alpha}+\Psi|^2 - 2k^2\left[\left(\Phi - \frac{1}{6}k^2\nu\right)\Psi^* + c.c.\right] + \left(2-K_{\mathrm{B}}\right)k^2\left[\left(\dot{\alpha}+\Psi\right)\chi^* + c.c.\right] - \left(2-K_{\mathrm{B}}\right)\left(1+\lambda_s\right)k^2|\chi|^2\Big\} \end{aligned} \tag{141}$$

where fields in (141) have a subscript $\vec{k}$ to explicitly show their k dependence as they are the Fourier modes of those in (140) and (c.c) means complex conjugate.

We now find the normal modes. It is sufficient to work in the Newtonian gauge by setting $\nu = \zeta = 0$. We set the time dependence of all perturbations to $e^{i\omega t}$ and rewrite (141) as $\int dt \int \frac{d^3k}{(2\pi)^3} Z^\dagger U Z + (h.c)$, where $Z = \{\Psi, \Phi, \alpha, \varphi\}$ and U is a 4×4 matrix of coefficients which depend on ω, k and the other AeST parameters. The determinant of U is found to be

$$\det U = 4k^6\omega^2\Big\{\left(2-K_{\mathrm{B}}\right)\left[(2+K_{\mathrm{B}}\lambda_s)k^2 + 2\mathcal{K}_2\mathcal{Q}_0^2(1+\lambda_s)\right] - 2\mathcal{K}_2K_{\mathrm{B}}\omega^2\Big\} \tag{142}$$

so setting $\det U = 0$ gives the two dispersion relations

$$\omega^2 = 0 \tag{143}$$
$$\omega^2 = c_s^2 k^2 + \mathcal{M}^2 \tag{144}$$

where the scalar speed of sound is

$$c_s^2 = \frac{(2-K_{\rm B})}{\mathcal{K}_2 K_{\rm B}}(1+\frac{1}{2}K_{\rm B}\lambda_s) \tag{145}$$

Notice that the first mode does not lead to a propagating wave but rather to a mode evolving as $\sim A_0 + B_0 t$ where A_0 and B_0 are k-dependent constants. Interestingly also, the second mode is massive with the same mass as the vector mode $\vec{\beta}$. Positivity of c_s^2 implies further stability conditions in addition to the ones found above for the vector modes. Specifically, since from (139) we have $\lambda_s > -1$, then $1 + \frac{1}{2}K_{\rm B}\lambda_s > 0$ leading to the condition

$$\mathcal{K}_2 > 0. \tag{146}$$

To further understanding the nature of the scalar modes a Hamiltonian analysis is performed. This allows the system to be 'de-constrained' by removing the redundant gauge and nondynamical degrees of freedom and leads to a characterization of the significance of the $\omega = 0$ normal mode. Starting from (141), notice that out of the six fields, two (Ψ and ζ) do not contain time derivatives. We determine the canonical momenta [E27] for the other four which are found to be

$$P_\Phi = -4\left(3\dot{\Phi} + k^2\zeta\right), \tag{147}$$
$$P_\nu = \frac{1}{3}k^4\left(\dot{\nu} + 2\zeta\right), \tag{148}$$
$$P_\chi = 4\mathcal{K}_2\left[\dot{\chi} - \mathcal{Q}_0(\dot{\alpha} + \Psi)\right], \tag{149}$$
$$\begin{aligned} P_\alpha = &-4\mathcal{K}_2\mathcal{Q}_0\dot{\chi} + 2\left(K_{\rm B}k^2 + 2\mathcal{K}_2\mathcal{Q}_0^2\right)(\dot{\alpha} + \Psi) \\ &+ 2\left(2 - K_{\rm B}\right)k^2\chi, \end{aligned} \tag{150}$$

and where χ has been used instead of φ as the dynamical variable. Performing a Legendre transformation, the Hamiltonian density is

$$\begin{aligned} \mathcal{H} = &-\frac{1}{24}|P_\Phi|^2 + \frac{3}{2k^4}|P_\nu|^2 + \frac{1}{8\mathcal{K}_2}|P_\chi|^2 + \frac{1}{4k^2K_{\rm B}}|P_\alpha + \mathcal{Q}_0P_\chi|^2 - 2k^2|\Phi - \frac{1}{6}k^2\nu|^2 + \frac{2-K_{\rm B}}{K_{\rm B}}k^2(2+K_{\rm B}\lambda_s)|\chi|^2 \\ &- \frac{2-K_{\rm B}}{2K_{\rm B}}\left[(P_\alpha + \mathcal{Q}_0P_\chi)\chi^* + (P_\alpha^* + \mathcal{Q}_0P_\chi^*)\chi\right] + C_\Psi\Psi^* + C_\Psi^*\Psi + C_\zeta^*\zeta + C_\zeta\zeta^* \end{aligned} \tag{151}$$

Since the variables Ψ and ζ are not dynamical, their function is to act as Lagrange multipliers imposing the constraints

$$C_\Psi \equiv 2k^2\Phi - \frac{k^4}{3}\nu - \frac{1}{2}P_\alpha \approx 0 \tag{152}$$
$$C_\zeta \equiv -P_\nu - \frac{k^2}{6}P_\Phi \approx 0 \tag{153}$$

which essentially cast ν and P_ν as functions of the other variables. As usual we use the symbol $\approx$ to denote weakly vanishing constraints (those that vanish only on-shell) [E5]. Notice also that the variable α is cyclic, therefore its canonical momentum P_α is conserved and is an integral of motion.

The next step of the Hamiltonian analysis is to ensure that the constraints are preserved by time evolution according to the Hamiltonian $H = \int \frac{d^3k}{(2\pi)^3}\mathcal{H}$. The Poisson brackets on phase space are defined as

$$\{f, g\} = (2\pi)^3 \int d^3k \left[\sum_I \left(\frac{\delta f}{\delta X^I}\frac{\delta g}{\delta P_{X^I}^*} - \frac{\delta g}{\delta X^I}\frac{\delta f}{\delta P_{X^I}^*}\right)\right] \tag{154}$$

where I runs over $\{\Phi, \nu, \chi, \alpha\}$. The time evolution of a variable f is $\dot{f} = \{f, H\}$, so we have

$$\dot{C}_\Psi = C_\zeta, \quad \dot{C}_\zeta = 0. \tag{155}$$

Hence, the constraints are preserved by time evolution on-shell. Therefore as one might expect, the stability of the primary constraints in the absence of gauge fixing does not create new constraints. Having ensured the stability of constraints in the Hamiltonian, we can now simplify the system by employing gauge fixing. In the Hamiltonian formulation, primary first-class constraints generate gauge transformations. The infinitesimal change of a phase space quantity f under this gauge transformation generated by the constraint C_I is given by:

$$\Delta f = \{f, C_I^*[\epsilon_I]\}\,. \tag{156}$$

where we have introduced the smearing $C_I^*[\epsilon_I]$ of a constraint C_I^* with test function ϵ_I defined as

$$C_I^*[\epsilon_I] \equiv \int \frac{d^3k}{(2\pi)^3}\epsilon_{I,\vec{k}}C^*_{I,\vec{k}} \tag{157}$$

Consider the following gauge transformations generated by the constraints C_ζ and C_Ψ:

$$\Delta\nu = \left\{\nu, C_\zeta^*[\epsilon_\zeta]\right\} = -\epsilon_\zeta \tag{158}$$

$$\Delta P_\nu = \{P_\nu, C_\Psi^*[\epsilon_\Psi]\} = \frac{1}{3}k^4\epsilon_\Psi \tag{159}$$

Thus, one may set ν and P_ν to zero by a gauge transformation by choosing $\epsilon_\zeta = \nu$ and $\epsilon_\Psi = -\frac{3}{k^4}P_\nu$. It can then be checked what constraints are placed on the Lagrange multipliers ζ, Ψ by this gauge fixing. Invoking two new gauge fixing constraints:

$$G_\nu \equiv \nu \approx 0 \tag{160}$$

$$G_{P_\nu} \equiv P_\nu \approx 0 \tag{161}$$

it follows that their time evolution is given by

$$\{G_\nu, H\} = \frac{3}{k^4}G_{P_\nu} - 2\zeta \tag{162}$$

$$\{G_{P_\nu}, H\} = \frac{2}{3}k^4\left(\Psi - \Phi\right) + \frac{1}{9}k^6G_\nu \tag{163}$$

Therefore the following gauge restrictions are placed on the Lagrange multipliers: $\zeta = 0$ and $\Psi = \Phi$. These conditions can be recognized, respectively, as a restriction to the conformal Newtonian gauge and the content of the Einstein equation here dictating equality between metric potentials in this gauge. These may be adopted conditions alongside the constraints G_ν, G_{P_ν} in the Hamiltonian (151) and the primary constraints, yielding in addition

$$P_\Phi \approx 0 \tag{164}$$

$$\Phi \approx \frac{1}{4k^2}P_\alpha \tag{165}$$

so that the deconstrained Hamiltonian density is

$$\begin{aligned}\mathcal{H}^{(\mathrm{Dec})} =& \frac{1}{8\mathcal{K}_2}|P_\chi|^2 + \frac{1}{4k^2K_{\mathrm{B}}}|P_\alpha + \mathcal{Q}_0P_\chi|^2 - \frac{1}{8k^2}|P_\alpha|^2 - \frac{2-K_{\mathrm{B}}}{2K_{\mathrm{B}}}\left[(P_\alpha + \mathcal{Q}_0P_\chi)\chi^* + (P_\alpha^* + \mathcal{Q}_0P_\chi^*)\chi\right] \\ &+ \frac{2-K_{\mathrm{B}}}{K_{\mathrm{B}}}k^2(2 + K_{\mathrm{B}}\lambda_s)|\chi|^2.\end{aligned} \tag{166}$$

The Hamiltonian density $\mathcal{H}^{(\mathrm{Dec})}$ is free of constraints but its form remains rather complicated. We can make an additional simplification by making a canonical transformation to canonical pairs (P_X, X), (P_Y, Y) defined via

$$\chi = \sqrt{\frac{K_{\rm B}k^2 + (2-K_{\rm B})\mu^2}{K_{\rm B}(2-K_{\rm B})}}\frac{\mathcal{Q}_0}{\mu k}X + \frac{1}{2}\frac{P_Y}{(2+K_{\rm B}\lambda_s)k^2 + (2-K_{\rm B})(1+\lambda_s)\mu^2} \tag{167}$$

$$P_\chi = \sqrt{\frac{K_{\rm B}(2-K_{\rm B})}{K_{\rm B}k^2 + (2-K_{\rm B})\mu^2}}\frac{\mu k}{\mathcal{Q}_0}\left[\frac{2(2-K_{\rm B})\mathcal{Q}_0}{K_{\rm B}}X + P_X\right] - \frac{1}{\mathcal{Q}_0}\frac{(2-K_{\rm B})(1+\lambda_s)\mu^2}{(2+K_{\rm B}\lambda_s)k^2 + (2-K_{\rm B})(1+\lambda_s)\mu^2}P_Y \tag{168}$$

$$\alpha = Y + \sqrt{\frac{K_{\rm B}(2-K_{\rm B})}{K_{\rm B}k^2 + (2-K_{\rm B})\mu^2}}\frac{\mu k}{\mathcal{Q}_0}\left[\frac{\mathcal{Q}_0}{K_{\rm B}k^2}X + \frac{1}{2}\frac{P_X}{(2+K_{\rm B}\lambda_s)k^2 + (2-K_{\rm B})(1+\lambda_s)\mu^2}\right] \tag{169}$$

$$P_\alpha = P_Y \tag{170}$$

where

$$\mu^2 \equiv \frac{2\mathcal{K}_2\mathcal{Q}_0^2}{2-K_{\rm B}} \tag{171}$$

This gives a Hamiltonian density

$$\tilde{\mathcal{H}} = \frac{1}{4}|P_X|^2 + \left(c_s^2k^2 + \mathcal{M}^2\right)|X|^2 + \frac{(2-K_{\rm B})^2\lambda_s}{16K_{\rm B}\mathcal{K}_2}\frac{1-\frac{k_*^2}{k^2}}{c_s^2k^2+\mathcal{M}^2}|P_Y|^2 \tag{172}$$

where

$$k_*^2 = \frac{1+\lambda_s}{\lambda_s}\mu^2 \tag{173}$$

It can be seen then that the system can be cast in terms of two decoupled fields, X and Y, with canonical momenta P_X and P_Y respectively, and each field corresponds to one of the normal modes in (144). Specifically, the field X propagates the massive modes in (144) while the field Y corresponds to the non-propagating $\omega = 0$ modes. One notices that the sign of the $|P_Y|^2$ term in (172) is not positive definite but rather depends on the relevant wave number k and parameters λ_s and k_*. Clearly as $k \to \infty$, $|P_Y|^2$ comes with a positive sign provided $\lambda_s > 0$, and negative otherwise, which provides an additional condition to the one found for vector modes in (139). Taking both scalar and vector mode conditions on the AeST parameters we require that

$$0 < K_{\rm B} < 2 \tag{174}$$

$$\mathcal{K}_2 > 0 \tag{175}$$

$$\lambda_s > 0 \tag{176}$$

These conditions also imply that $G_{\rm N} > \tilde{G}$ always. More generally, when $k > k_*$ defined by (173), the Hamiltonian density is positive while when $k < k_*$, negative Hamiltonian density can occur if the $|P_Y|^2$ term in (172) becomes significant. The solutions for $\omega = 0$ correspond to $Y = A_0(\vec{k})t + B_0(\vec{k})$ while $P_Y = A_0(\vec{k})$. Thus the mode which could cause negative Hamiltonian densities is the one evolving linearly with t. Such instabilities are likely akin to Jeans-type instabilities and do not cause quantum vacuum instability at low momenta [E89]. As discussed in [H4], for a spherically symmetric static source of mass M, the transition between the MOND and an oscillatory μ-dominated regime occurs at $r_C \sim \left(r_M\mu^{-2}\right)^{1/3}$ where $r_M \sim \sqrt{\frac{G_{\rm N}M}{a_0}}$ is the MOND scale which signifies the transition between the Newtonian and MOND regimes on even smaller distances. Thus, on observational grounds μ^{-1} must be larger than $\sim$ Mpc, otherwise, the MOND regime would not occur at the scales of galaxies at distances $\sim$ kpc (for the Milky Way $r_M \sim$ 8kpc). A system with a MOND scale of $\sim$ Mpc occurs if its mass is $\sim 10^{15}M_\odot$ which is much larger than typical masses of bound structures. Thus, for $\lambda_s \geq 1$, the scale k_* is always hidden inside the MOND regime (i.e. $k_* < r_M^{-1}$) so that the negative Hamiltonian does not occur in the GR limit for all systems of interest. At smaller wave numbers $< r_M^{-1}$, AeST enters the MOND regime (in which case $\lambda_s = 0$) which would signify that the Y-mode always has a negative Hamiltonian. However, then there exists a higher order term $\sim |\mathcal{Y}|^{3/2}/a_0 = |\vec{\nabla}\chi|^3/a_0$

that is not part of the analysis above, and which may stabilize the system To investigate this, we set $\lambda_s = 0$ in the expansion (178) and add the MOND term

$$\mathcal{J}_{NL} = \frac{2\lambda_s}{3(1+\lambda_s)a_0}|\mathcal{Y}|^{3/2} \tag{177}$$

where the presence of λ_s above is a relic of its influence on the observed value of Newton's constant in the strong gravity regime so that

$$\mathcal{F} = (2 - K_{\rm B})\mathcal{J}_{NL}(\mathcal{Y}) - 2\mathcal{K}_2\left(\mathcal{Q} - \mathcal{Q}_0\right)^2 + \ldots \tag{178}$$

With this, the scalar mode action turns into

$$S^{(\mathrm{S},new)} = S^{(\mathrm{S})} - (2 - K_{\rm B})\int d^4x \mathcal{J}_{NL} \tag{179}$$

where $S^{(\mathrm{S})}$ is given by (140). Since $\mathcal{J}_{NL}$ does not contain any time derivatives, the canonical momenta remain the same as those from the analysis of linear perturbations. Thus the Hamiltonian analysis of the previous sections follows through so that the deconstrained Hamiltonian is

$$\tilde{\mathcal{H}}^{(new)} = \tilde{\mathcal{H}} + \frac{2(2-K_{\rm B})\lambda_s}{3(1+\lambda_s)a_0}|\vec{\nabla}\chi|^3 \tag{180}$$

where $\tilde{\mathcal{H}}$ is given by (172) (with $\lambda_s = 0$) and χ is given by (167). Observe that the nonlinear MOND term above comes with a positive sign and also, it will dominate $\tilde{\mathcal{H}}^{(new)}$ for large χ. Thus, it is suggestive that the MOND term may make $\tilde{\mathcal{H}}^{(new)}$ to be bounded from below. Indeed, that turns out to be the case for wave numbers $k > \mu$, as is shown in detail in an appendix of [H5]. At smaller wave numbers $k < \mu$ the MOND term is not sufficient to make the Hamiltonian bounded from below, however, that is the regime where the Minkowski approximation is expected to break down and expanding on FRW (or even more specifically de Sitter) is more appropriate.

Clearly a Hamiltonian analysis of linear scalar perturbations in the AeST model around Minkowski space has been productive. In the next section, the non-perturbative Hamiltonian formulation of the AeST model is presented.

5.5.4 Aether scalar tensor theory: Hamiltonian Formalism

The casting of the full AeST theory into Hamiltonian form yields a number of advantages. The casting of the model's equations of motion in the form of Hamilton's first-order equations of motion puts the equations of motion in a form more amenable to numerical solution which is a vital part of modern gravitational theory, where models are applied to complex situations where analytic solutions are not possible. The completion of the canonical analysis also enables clarification of other issues, such as, the number of degrees of freedom that the model possesses and whether the model is an example of an irregular system, that is, a theory where the canonical structure varies throughout phase space. A manifestation of the latter can be that perturbations around some backgrounds describe different number of degrees of freedom than perturbations around others, potentially indicating an instability of that background. This can be a fatal result for a theory if that background is one that is expected to approximate the geometry of our universe on long time scales.

As a necessary first step towards constructing the Hamiltonian formalism for the AeST model, we must make a distinction between space and time. Specifically, we assume that for the region of spacetime of interest, there exists a global time coordinate $t(x^\mu)$. Given this, we may define a 'flow of time' vector field t^μ which satisfies $t^\mu\nabla_\mu t = 1$. We use the notation $\dot{f} \equiv \partial_t f$ for some field f. Furthermore, we may define a vector field n^μ which is normal to surfaces of constant t; as such, this field is timelike and may be defined so that it has unit-norm i.e. $g_{\mu\nu}n^\mu n^\nu = -1$. We may expand this time field as $t^\mu = Nn^\mu + N^\mu$, where we have introduced the lapse function N and shift vector N^μ, which are given respectively by $N = -t^\mu n_\mu$ and $N_\mu = q_{\mu\nu}t^\nu$. We coordinatize surfaces of constant t by spatial coordinates x^i, where i, j, k will be used throughout to denote spatial coordinate indices. The full spacetime metric may be decomposed as:

$$g_{\mu\nu} = -n_\mu n_\nu + q_{\mu\nu}, \tag{181}$$

where $q_{\mu\nu}$ is the metric on the spatial hypersurface (and therefore, for example $q_{\mu\nu}n^{\mu} = 0$). It is also useful to introduce the following tensor:

$$\hat{q}_{\mu\nu} \equiv g_{\mu\nu} + A_{\mu}A_{\nu} \tag{182}$$

Note the difference between $\hat{q}_{\mu\nu}$ defined in (182) and $q_{\mu\nu}$ defined in (181). It is useful to define a spatial derivative $\hat{\partial}_{\mu} = {q_{\mu}}^{\nu}\partial_{\nu}$ and covariant derivative D_{μ} compatible wiht $q_{\mu\nu}$, i.e. $D_{\alpha}q_{\mu\nu} = 0$. Specifically:

$$D_{\mu}q_{\alpha\beta} = \hat{\partial}_{\mu}q_{\alpha\beta} - \gamma^{\nu}_{\mu\alpha}g_{\nu\beta} - \gamma^{\nu}_{\mu\beta}g_{\alpha\nu} = 0 \tag{183}$$

where $\gamma^{\mu}_{\alpha\beta}$ are Levi-Civita symbols associated with the metric $q_{\mu\nu}$ and derivative $\hat{\partial}_{\mu}$. Given $q_{\mu\nu}$ and n^{μ}, a further useful quantity is extrinsic curvature tensor $K_{\mu\nu}$, defined as

$$K_{\mu\nu} \equiv \frac{1}{2}\mathcal{L}_n q_{\mu\nu} = {q_{\mu}}^{\alpha}\nabla_{\alpha}n_{\nu} \tag{184}$$

In component form, we need

$$K_{ij} = \frac{1}{2N}\left(\dot{q}_{ij} - D_i N_j - D_j N_i\right) \tag{185}$$

while the components of the metric, n^{μ} and the Christoffel connection are displayed in an appendix of [H7]. We will adhere to the convention that spatial indices are always lowered and raised with the spatial metric q_{ij}, i.e. ${K^i}_j = q^{ik}K_{kj}$. For the vector field $\hat{A}_{\mu}$ we consider a similar decomposition

$$A_{\mu} = \chi n_{\mu} + A_{\mu} \tag{186}$$

where $A_{\mu} \equiv {q_{\mu}}^{\nu}A_{\nu}$ and

$$\chi = -n^{\mu}A_{\mu} \tag{187}$$

We now present the necessary steps in writing the action (108) in $3+1$ form. One of the steps involves solving for $\dot{\phi}$ in terms of the canonical momenta $\delta S/\delta\dot{\phi}$ and this will involve having to invert potentially very complicated combinations of functions $\partial\mathcal{F}/\partial\mathcal{Y}$ and $\partial\mathcal{F}/\partial\mathcal{Q}$. Instead we can move this structure into elsewhere in the theory by introducing auxiliary fields μ and ν such that we set

$$\mathcal{F}(\mathcal{Y},\mathcal{Q}) = -\nu\mathcal{Q}^2 + \mu\mathcal{Y} + \mathcal{U}(\nu,\mu) \tag{188}$$

For the scalar field ϕ, we then find the scalars $\mathcal{Q}$ and $\mathcal{Y}$ as

$$\mathcal{Q} = \chi\sigma + A^i D_i\phi \tag{189}$$

$$\mathcal{Y} = |\vec{A}|^2\sigma^2 + 2\chi\sigma A^i D_i\phi + \left(q^{ij} + A^i A^j\right)D_i\phi D_j\phi \tag{190}$$

where we have defined

$$\sigma = \frac{1}{N}\left(\dot{\phi} - N^i D_i\phi\right) \tag{191}$$

It is additionally useful to define following decomposition of the tensor $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$:

$$F_{ij} \equiv 2D_{[i}A_{j]} = F_{ij} \tag{192}$$

and

$$F_i \equiv \frac{1}{N}F_{0i} = \frac{1}{N}\left[\dot{A}_i + D_i\left(N\chi - N^j A_j\right)\right], \tag{193}$$

we define the "magnetic" aspect of A^i as

$$B^k = \frac{1}{2}\epsilon^{kij}F_{ij}, \tag{194}$$

with inverse $F_{ij} = \epsilon_{ijk} B^k$ and the "electric" aspect of A^i as

$$E_i = F_i + \frac{1}{N}\epsilon_{ijk} N^j B^k. \tag{195}$$

Then, it can be shown that the 3+1 form of the action (108) is

$$\begin{aligned}
S = \int d^4x \frac{N\sqrt{q}}{16\pi\tilde{G}} \Big\{ & \mathcal{R} + |K^2| - |K|^2 - 2\Lambda + K_{\mathrm{B}}\left(|\vec{E}|^2 - |\vec{B}|^2\right) + 2(2-K_{\mathrm{B}})\sigma\vec{A}\cdot\vec{E} \\
& + 2(2-K_{\mathrm{B}})\left(\chi\vec{E} - \vec{A}\times\vec{B}\right)\cdot\vec{D}\phi + \nu\left(\chi\sigma + \vec{A}\cdot\vec{D}\phi\right)^2 \\
& - (2-K_{\mathrm{B}}+\mu)\left[|\vec{A}|^2\sigma^2 + 2\chi\sigma\vec{A}\cdot\vec{D}\phi + |\vec{D}\phi|^2 + \left(\vec{A}\cdot\vec{D}\phi\right)^2\right] - \mathcal{U}(\nu,\mu)\Big\} + S_m[g]
\end{aligned} \tag{196}$$

where $\mathcal{R}$ is the Ricci scalar corresponding to the spatial metric q_{ij}. Note that S is a functional of the fields $(q_{ij}, A_i, \phi, \mu, \nu, N, N^i)$. Passage to the canonical formulation of (108) is now possible. The next step in passing to the Hamiltonian formulation is to determine the canonical momenta which are

$$\Pi^{ij} \equiv \frac{\delta S}{\delta \dot{q}_{ij}} = \frac{\sqrt{q}}{16\pi\tilde{G}}\left(K^{ij} - Kq^{ij}\right) \tag{197}$$

$$\Pi^{i} \equiv \frac{\delta S}{\delta \dot{A}_{i}} = \frac{\sqrt{q}}{8\pi\tilde{G}}\left[K_{\mathrm{B}}E^i + (2-K_{\mathrm{B}})\left(\sigma A^i + \chi D^i\phi\right)\right] \tag{198}$$

$$\begin{aligned}
\Pi \equiv \frac{\delta S}{\delta\dot{\phi}} = \frac{\sqrt{q}}{8\pi\tilde{G}}\Big[& (2-K_{\mathrm{B}})\vec{A}\cdot\vec{E} + \nu\sigma \\
& - (2-K_{\mathrm{B}}+\mu-\nu)\,\vec{A}\cdot\left(\sigma\vec{A} + \chi\vec{D}\phi\right)\Big]
\end{aligned} \tag{199}$$

Letting $\hat{\Pi} \equiv \Pi^{ij}q_{ij}$ the inverse relations are

$$K^{ij} = \frac{16\pi\tilde{G}}{\sqrt{q}}\left(\Pi^{ij} - \frac{1}{2}\hat{\Pi}q^{ij}\right) \tag{200}$$

$$\begin{aligned}
\Xi\sigma = & \frac{8\pi\tilde{G}}{\sqrt{q}}\left[\Pi - \frac{2-K_{\mathrm{B}}}{K_{\mathrm{B}}}\vec{A}\cdot\vec{\Pi}\right] \\
& + \left(2\frac{2-K_{\mathrm{B}}}{K_{\mathrm{B}}} + \mu - \nu\right)\chi\vec{A}\cdot\vec{D}\phi
\end{aligned} \tag{201}$$

$$K_{\mathrm{B}}E^i = \frac{8\pi\tilde{G}}{\sqrt{q}}\Pi^i - (2-K_{\mathrm{B}})\left(\sigma A^i + \chi D^i\phi\right) \tag{202}$$

where

$$\Xi = \chi^2\nu - \left(2\frac{2-K_{\mathrm{B}}}{K_{\mathrm{B}}} + \mu\right)|\vec{A}|^2 \tag{203}$$

while the canonical momenta for μ and ν are identically zero:

$$\Pi^{(\mu)} \equiv \frac{\delta S}{\delta\dot{\mu}} \approx 0 \tag{204}$$

$$\Pi^{(\nu)} \equiv \frac{\delta S}{\delta\dot{\nu}} \approx 0 \tag{205}$$

Using (200), (201) and (202) to remove K_{ij}, E_i and σ from (196) leads to the Hamiltonian form of the action,

$$S = \int d^4x \Big\{ \Pi^{ij}\dot{q}_{ij} + \Pi^i\dot{A}_i + \Pi\dot{\phi} + \Pi^{(\mu)}\dot{\mu} + \Pi^{(\nu)}\dot{\nu} - N\mathcal{H} - N^i\mathcal{H}_i - \lambda^{(\mu)}\Pi^{(\mu)} - \lambda^{(\nu)}\Pi^{(\nu)} \Big\}, \tag{206}$$

where we have added Lagrange multiplier fields $\lambda^{(\mu)}$ and $\lambda^{(\nu)}$ which impose the constraints (204) and (205) via their equations of motion and where

$$\mathcal{H}_i = -2D_j\Pi^j{}_i + \Pi D_i\phi - \vec{D}\cdot\vec{\Pi}A_i - \epsilon_{ijk}\Pi^j B^k + \Pi^{(\mu)}D_i\mu + \Pi^{(\nu)}D_i\nu \tag{207}$$

is the diffeomorphism constraint and

$$\begin{aligned}\mathcal{H} =& \frac{8\pi\tilde{G}}{\sqrt{q}}\left[2\Pi^{ij}\Pi_{ij} - \hat{\Pi}^2 + \frac{1}{2K_{\rm B}}|\vec{\Pi}|^2 + \frac{C_1^2}{2\Xi}\right] + \chi\left[\frac{C_1C_2}{\Xi}\vec{A}\cdot\vec{D}\phi + \vec{D}\cdot\vec{\Pi} - \frac{2-K_{\rm B}}{K_{\rm B}}\vec{\Pi}\cdot\vec{D}\phi\right] \\ &+ \frac{\sqrt{q}}{16\pi\tilde{G}}\Bigg\{-\mathcal{R} + 2\Lambda + K_{\rm B}|\vec{B}|^2 + \left[\frac{C_2^2\chi^2}{\Xi} + 2 - K_{\rm B} + \mu - \nu\right]\left[\vec{A}\cdot\vec{D}\phi\right]^2 + 2(2-K_{\rm B})\vec{A}\times\vec{B}\cdot\vec{D}\phi \\ &+ \left[2 - K_{\rm B} + \mu + \frac{(2-K_{\rm B})^2}{K_{\rm B}}\chi^2\right]|\vec{D}\phi|^2 + \mathcal{U}\Bigg\},\end{aligned} \tag{208}$$

the Hamiltonian constraint. For compactness of notation have defined

$$C_1 \equiv \Pi - \frac{2-K_{\rm B}}{K_{\rm B}}\vec{A}\cdot\vec{\Pi}, \tag{209}$$

$$C_2 \equiv 2\frac{2-K_{\rm B}}{K_{\rm B}} + \mu - \nu. \tag{210}$$

Note that to arrive at the action (206) we have defined the coefficients multiplying $(\Pi^{(\mu)}, \Pi^{(\nu)})$ to be $(\lambda^{(\mu)} + N^iD_i\mu, \lambda^{(\nu)} + N^iD_i\nu)$ which we are free to do at the beginning of the constraint analysis. This leads to the 2nd line in (207). The combination

$$\mathcal{H}_{\rm pri} = N\mathcal{H} + N^i\mathcal{H}_i + \lambda^{(\mu)}\Pi^{(\mu)} + \lambda^{(\nu)}\Pi^{(\nu)} \tag{211}$$

is the *primary* Hamiltonian density and it is a sum of constraints on the phase space which is coordinatized by $\{q_{ij}, \Pi^{ij}, A_i, \Pi^i, \varphi, \Pi, \mu, \Pi^{(\mu)}, \nu, \Pi^{(\nu)}\}$. These constraints are obtained by varying (206) with $\{N, N^i, \lambda^{(\mu)}, \lambda^{(\nu)}\}$ and are given respectively by:

$$\mathcal{H} \approx 0 \tag{212}$$

$$\mathcal{H}_i \approx 0 \tag{213}$$

$$\Pi^{(\mu)} \approx 0 \tag{214}$$

$$\Pi^{(\nu)} \approx 0 \tag{215}$$

where $\approx$ denotes that these phase space functions need only vanish on the submanifold of phase space that they define ('weakly vanish'). The next step is to check whether these constraints are preserved by the time evolution generated by $\mathcal{H}_{\rm pri}$. It is found that for general N, $\lambda^{(\mu)}$ and $\lambda^{(\nu)}$, the primary constraints are preserved if, further, the following *secondary* constraints hold:

$$\mathcal{S}^{(\mu)} \equiv \frac{\partial\mathcal{H}}{\partial\mu} = \frac{\sqrt{q}}{16\pi\tilde{G}}\left(\mathcal{Y} + \frac{\partial\mathcal{U}}{\partial\mu}\right) \approx 0 \tag{216}$$

and

$$\mathcal{S}^{(\nu)} \equiv \frac{\partial\mathcal{H}}{\partial\nu} = \frac{\sqrt{q}}{16\pi\tilde{G}}\left(-\mathcal{Q}^2 + \frac{\partial\mathcal{U}}{\partial\nu}\right) \approx 0 \tag{217}$$

Equations (216) and (217) correspond to the Euler-Lagrange equations for the auxiliary fields μ and ν. In other words, given a prescribed $\mathcal{U}(\mu,\nu)$ one can use (216) and (217) to determine $\mu(\mathcal{Q},\mathcal{Y})$ and $\nu(\mathcal{Q},\mathcal{Y})$, where $\mathcal{Q}$ and $\mathcal{Y}$ are to be evaluated in phase space. Using (201) followed by (190) and (189) to collect terms together, one finds that

$$\mathcal{Q} = \frac{1}{\Xi}\left[\left(2\frac{2-K_{\rm B}}{K_{\rm B}} + \mu\right)\vec{A}\cdot\vec{D}\phi + \frac{8\pi\tilde{G}}{\sqrt{q}}\chi C_1\right] \tag{218}$$

	Primary				Secondary	
	$\mathcal{H}_i$	$\mathcal{H}$	$\Pi^{(\mu)}$	$\Pi^{(\nu)}$	$\mathcal{S}^{(\mu)}$	$\mathcal{S}^{(\nu)}$
$\mathcal{H}_i$	$\mathcal{H}_i$	$\mathcal{H}$	$\Pi^{(\mu)}$	$\Pi^{(\nu)}$	$\mathcal{S}^{(\mu)}$	$\mathcal{S}^{(\nu)}$
$\mathcal{H}$		$\mathcal{H}_i$	$\mathcal{S}^{(\mu)}$	$\mathcal{S}^{(\nu)}$	$U^{(\mu)}$	$U^{(\nu)}$
$\Pi^{(\mu)}$			0	0	$C^{(\mu)(\mu)}$	$C^{(\nu)(\mu)}$
$\Pi^{(\nu)}$				0	$C^{(\mu)(\nu)}$	$C^{(\nu)(\nu)}$
$\mathcal{S}^{(\nu)}$					$E^{(\mu)(\mu)}$	$E^{(\mu)(\nu)}$
$\mathcal{S}^{(\mu)}$						$E^{(\nu)(\nu)}$
	First Class	Second Class				

Table 1: Table of constraints and their Poisson Brackets, showing whether they vanish strongly, weakly, or not at all. The classification into primary/secondary and first/second class is marked. Note also that the combination $H_{\rm FC}$ defined through (223) is first class, with $\lambda^{(A)}$ being functions of all the phase space variables as determined through (221), even though some individual parts of $H_{\rm FC}$ are second class.

and

$$\mathcal{Y} = |\vec{D}\phi|^2 + \left(\vec{A}\cdot\vec{D}\phi\right)^2 + \frac{|\vec{A}|^2}{\Xi^2}\left[\frac{8\pi\tilde{G}}{\sqrt{q}}C_1 + C_2\chi\vec{A}\cdot\vec{D}\phi\right]^2 + \frac{2\chi}{\Xi}\left(\frac{8\pi\tilde{G}}{\sqrt{q}}C_1 + C_2\chi\vec{A}\cdot\vec{D}\phi\right)\vec{A}\cdot\vec{D}\phi, \qquad (219)$$

respectively. This procedure then reconstructs $\mathcal{F}(\mathcal{Q},\mathcal{Y})$ through (188) in terms of phase space variables with the help of (218) and (219).

Having found the secondary constraints (216) and (217), the analysis is not necessarily finished. We now define the secondary Hamiltonian through

$$H_{\rm sec} = H_{\rm pri} + \int d^3x\left[u^{(\mu)}\mathcal{S}^{(\mu)} + u^{(\nu)}\mathcal{S}^{(\nu)}\right] \qquad (220)$$

where $u^{(\mu)}$ and $u^{(\nu)}$ are Lagrange multipliers enforcing the secondary constraints $\mathcal{S}^{(\mu)}$ and $\mathcal{S}^{(\nu)}$ respectively. We then check that all constraints (primary and secondary) are preserved in time. Defining the indices $A, B, = \{(\mu),(\nu)\}$, it is found that preservation in time of primary constraints is secured if $u^A = 0$ whereas the preservation of secondary constraints implies the following equations:

$$F^A + \sum_B C^{AB}\lambda^{(B)} \approx 0 \qquad (221)$$

where

$$C^{AB} \equiv \frac{\partial\mathcal{S}^{(B)}}{\partial A} = \begin{pmatrix} \frac{\partial\mathcal{S}^{(\mu)}}{\partial\mu} & \frac{\partial\mathcal{S}^{(\mu)}}{\partial\nu} \\ \frac{\partial\mathcal{S}^{(\nu)}}{\partial\mu} & \frac{\partial\mathcal{S}^{(\nu)}}{\partial\nu} \end{pmatrix} \qquad (222)$$

and F^A is a function of phase space fields and the Lagrange multiplier field N. If the matrix C^{AB} is invertible then the two equations above *determine* the Lagrange multipliers λ^A which then become functions of all the phase-space variables. Then the Hamiltonian analysis is complete and no further constraints in phase-space are required, with the conclusion that the theory possesses three first class primary constraints $\mathcal{H}_i$, three second class primary constraints $\mathcal{H}$ and $\Pi^{(A)}$, and two second class secondary constraints $\mathcal{S}^{(A)}$. We list all the constraints and their Poisson brackets in Table 1. On the other hand, if the matrix has a vanishing determinant, there exists a left null eigenvector which when applied to the consistency relation may produce further constraints, called *tertiary* constraints. Given the forms of (218),(219) and equations of motion (216),(217) it is to be expected that C^{AB} is indeed invertible in general situations. Finally we note that the determination of all constraints and their classification enables the construction of the Dirac bracket [E25].

Having found all the constraints and solved for the Lagrange multipliers, we may now form the *first class* Hamiltonian H_{FC}. We find this as the secondary Hamiltonian with the Lagrange multipliers substituted in for. Given that $u^{(A)} \approx 0$, we have that $H_{\mathrm{sec}} \approx H_{\mathrm{pri}}$, hence,

$$H_{\mathrm{FC}} = \int d^3x \left[N\mathcal{H} + N^i\mathcal{H}_i + \lambda^{(\mu)}\Pi^{(\mu)} + \lambda^{(\nu)}\Pi^{(\nu)} \right] \tag{223}$$

with $\lambda^{(\mu)}$ and $\lambda^{(\nu)}$ being functions of the phase-space variables $\{q_{ij}, A_i, \phi, \Pi^{ij}, \Pi^i, \Pi, \mu, \nu\}$, that is, $\Pi^{(\mu)}$ and $\Pi^{(\nu)}$ are absent from $\lambda^{(A)}$.

To summarize, this analysis has revealed the existence of four first class constraints and four second class constraints. The first class constraints consist of the three (primary) constraints $\mathcal{H}_i$ defined in (207) and the first class Hamiltonian H_{FC} defined in (223), which is the linear combination of the primary Hamiltonian constraint $\mathcal{H}$ defined in (208) (by itself second class), $\mathcal{H}_i$ and also $\Pi^{(\mu)}$ and $\Pi^{(\nu)}$. The four second class constraints are the two canonical momenta $\Pi^{(\mu)}$ and $\Pi^{(\nu)}$ of the auxiliary fields μ and ν, see (204) and (205), and the two secondary constraints $\mathcal{S}^{(\mu)}$ and $\mathcal{S}^{(\nu)}$ defined through (216) and (217). The existence of these second class constraints arises from the presence of the auxiliary fields μ and ν. See table 1 for a summary of these constraints.

We may use the constraint analysis to count the number of physical degrees of freedom. We have six variables in the spatial metric q_{ij}, three in A_i and one for each of ϕ, μ and ν, that is 12 in total. Counting in the canonical momenta doubles this to 24. We subtract the four second class constraints and twice the number of first class constraints which remove the gauge redundant degrees of freedom, that is, we subtract 12 degrees of freedom because of the constraints. We finally divide by two to find six physical degrees of freedom.

As a useful check on the results, the full Hamiltonian was expanded to quadratic order in perturbations around Minkowski space and it was shown to lead to the same results as found in [H5] using slightly different methods. In the process, it was shown that the number of perturbative degrees of freedom found in [H5] matches the number found here using the full non-linear theory. The non-perturbative formalism may be used to compute the quadratic Hamiltonian of AeST theory on other backgrounds in order to determine whether those backgrounds are stable or not. Of particular interest are the cases of de Sitter space (relevant for initial investigation of the effect of non-vanishing background curvature on perturbations) and static spherically symmetric configurations (relevant for questions such as the stability of astrophysical systems in this model) which will be a topic of future work.

5.5.5 Further work

The AeST model has attracted interest from other researchers, with notable results being found regarding potential constraints on the theory via limits on the emission of Cherenkov radiation by stars in the model [E120] and constraints placed on the theory via data from weak gravitational lensing [E121].

I have investigated additional solutions to the AeST model. In a dynamical systems analysis of a number of AeST models in FRW symmetry was conducted [H9]. Notably, it was shown that a class of models exist which avoid the tension of models satisfying (107) by instead producing an effective fluid (defined as the ratio between fluid pressure and density) with equation of state that satisfies $w \to 0$ at early cosmological times. This new class of models can also fit cosmological data without DM.

6 Presentation of teaching, organisational, and ‘popularisation of science’ achievements

6.1 Teaching achievements

Academic teaching

- 2012 – 2014: Teaching (via tutorials) of undergraduate physics students at Imperial College, London to prepare them for their ‘Comprehensive’ exams. These are exams of extremely broad scope, so the role is rather demanding.

Recent Invited lectures

- *'The concept of Torsion in gravitational physics'*, University of Tartu, Estonia (2019)
- *'A new relativistic theory for Modified Newtonian Dynamics'*, Queen Mary University London, United Kingdom (2021)
- *'Dark matter or modified gravity?'*, University of Gdańsk, Poland, and Chinese Academy of Sciences Beijing China (2022)
- *'Is Newtonian dynamics modified?'*, University of Oxford , United Kingdom (2022)
- *'Gravity and Cartan geometry'*, Jagiellionian University Kraków, Poland (2022)
- *'Gravity and Parameterized Field Theories'*, Czech Academy of Sciences Prague, Czechia (2023)
- *'An introduction to gravitation and cosmology'*, Xiamen University, Malaysia (2023)

6.2 Organizational achievements

- 2016 – 2019: Assistance in planning of presence of the Czech Academy of Science at the Prague Science Fair. I also participated in this event which consisted of engaging members of the public and helping them with any questions about physics that they might have.

6.3 Popularisation of science achievements

- Discussion with journalists which contributed towards coverage of Model A in *New Scientist* and Model B in *The Atlantic* and *Science* magazines.
- Public lecture *'An Introduction to gravitation'*, presented at the High School 3 in Gdynia, February 2024 (Around 120 attendees)
- Public lecture: *'An Introduction to spacetime and gravity'* presented at the University of Gdańsk to local schoolchildren (Around 160 attendees)

7 Other scientific achievements

7.1 Supervision of Students

- 2022 – Present Day: Co-supervision with Professor Paweł Horodecki of doctoral student Mehraveh Nikjoo.
- 2023: Supervision of undergraduate student Khai Shuen Ng who was visiting the University of Gdańsk from Xiamen University Malaysia on a research internship project. The scope of the project was to explore a recently proposed 'gravitational dipole' solution to the system of GR coupled to a massless scalar field.
- 2023: Supervision of student Toh Yu Xuan who was visiting the University of Gdańsk from Xiamen University Malaysia on a research internship project. The scope of the project was to develop techniques towards developing an action principle for the Newton-Cartan theory of gravity.
- 2023: Supervision of University of Gdańsk undergraduate students Marek Majoch and Wiktoria Borkowska on a research internship project. The scope of the project was to determine the change of appearance of the night sky under the Lorentz transformations of special relativity.

7.2 Awards and Memberships of Scientific Organizations

- I am the Principal Investigator of Polonez BIS fellowship No. 2021/43/P/ST2/02141 co-funded by the Polish National Science Centre and the European Union Framework Programme for Research and Innovation Horizon 2020 under the Marie Skłodowska-Curie grant agreement No. 945339.
- I am a member of the Polish Society on Relativity (Polskie Towarzystwo Relatywistyczne)

7.3 Track record before PhD

The main focus of my PhD was on the cosmological consequences of vector tensor models of modified gravity - i.e. theories coupling the metric tensor to a single vector field A^{μ}. In [O2] it was shown that Bekenstein's TeVeS theory [E43] - which ostensibly takes the form of a bimetric theory with scalar and vector degrees of freedom - can be cast as a vector tensor theory. In [O5] a vector tensor model with non-canonical kinetic terms (I proposed this model) was shown to be the first relativistic theory of MOND that in the appropriate limit reproduces the original proposal by Milgrom rather than a multi-field generalization thereof. In [O6] solar system constraints on this model were considered whilst in [O7] the scope for the model to explain the growth of large scale structure in cosmology in the absence of DM was explored. In [O3], an exploration of FRW solutions to a self consistent effective field theory of modified gravity was conducted. In [O8, O10], a comprehensive survey of constraints from cosmological data on the Einstein-Aether vector tensor model and some generalizations of the model from cosmological data [E38].

Additional research involved a characterization of FRW solutions to Milgrom's bimetric theory of modified gravity [O9] and the role of orthographic correlations in astrophysics [O11].

7.4 Additional track record after PhD

As follows is a selection of research areas not included in the habilitation series that I have contributed to following the completion of my PhD:

On the possibility of anisotropic curvature in cosmology [O12] In addition to shear and vorticity a homogeneous background may also exhibit anisotropic curvature. In this publication a class of spacetimes was shown to exist where the anisotropy is solely of the latter type, and the shear-free condition is supported by a canonical, massless 2-form field. My main contribution was the discovery that the system of such a field coupled to GR contains these spacetimes as solutions. Such spacetimes possess a preferred direction in the sky and at the same time a CMB which is isotropic at the background level. A distortion of the luminosity distances is derived and used to test the model against the CMB and supernovae (using the Union catalog), and it is concluded that the latter exhibit a higher-than-expected dependence on angular position.

Cosmology with a spin [O13] Using the chiral representation for spinors a particularly transparent way to generate the most general spinor dynamics in a theory where gravity is ruled by the Einstein-Cartan-Holst action is presented. In such theories torsion need not vanish, but it can be reinterpreted as a four-fermion self-interaction within a torsion-free theory. The self-interaction may or may not break parity invariance, and may contribute positively or negatively to the energy density, depending on the couplings considered. Cosmological models ruled by a spinorial field were considered within this theory and it was found that while there are cases for which no significant cosmological novelties emerge, the self-interaction can also turn a mass potential into an upside-down Mexican hat potential. Then, as a general rule, the model leads to cosmologies with a bounce, for which there is a maximal energy density, and where the cosmic singularity has been removed. These solutions are stable, and range from the very simple to the very complex. My main contribution was a) the proposal of the novel couplings between fermions and the gravitational field that allowed the existence of this variety of cosmological solutions and b) co-analysis with collaborators Professors Tom Kibble and Joao Magueijo of said solutions.

A first-order approach to conformal gravity [O16] This article investigates whether a spontaneously-broken gauge theory of the group $SU(2,2)$ may be a viable alternative to GR. The basic ingredients of the theory are an gauge field and a Higgs field W in the adjoint representation of the group with the Higgs field producing the symmetry breaking . The action for gravity is polynomial in and the field equations are first-order in derivatives of these fields. The new symmetry in the gravitational sector may be interpreted in terms of an emergent local scale symmetry and the existence of 'conformalized' GR and fourth-order Weyl conformal gravity as limits of the theory was demonstrated. Maximally symmetric spacetime solutions to the full theory are found and stability of the theory around these solutions was investigated; it was shown that regions of the theory's parameter space describe perturbations identical to that of GR coupled to a massive scalar field and a massless one-form field. The coupling of gravity to matter was considered and it was shown that Lagrangians for all fields are naturally gauge-invariant, polynomial in fields and yield first-order field equations; no auxiliary fields are introduced. Familiar Yang–Mills and Klein–Gordon type Lagrangians are recovered on-shell in the general-relativistic limit of the theory. In this formalism, the general-relativistic limit coincides with a spontaneous breaking of scale invariance and it was shown that this generates mass terms for Higgs and spinor fields. Though my collaborator Dr Hans Westman had some input to this article, nearly all of the above content was proposed and carried about by myself.

Parity violating Friedmann Universes [O19] This article considers extensions of the Einstein-Cartan theory where the cosmological constant Λ is promoted to a field, at the cost of allowing for torsion even in the absence of spinors. It was found that some standard notions about FRW universes collapse in these theories, most notably that spatial homogeneity and isotropy may now coexist with violations of parity invariance. The parity-violating solutions have non-vanishing Weyl curvature even within FRW models. The presence of parity-violating torsion opens up the space of possible such theories with relevant FRW modifications; in particular the Pontryagin term can play an important role even in the absence of spinorial matter. The article presents a number of parity-violating solutions with and without matter. The former are the non-self-dual vacuum solutions long suspected to exist. The latter lead to tracking and non-tracking solutions with a number of observational problems, unless we invoke the Pontryagin term. An examination of the Hamiltonian structure of the theory reveals that the parity-even and the parity-violating solutions belong to two distinct branches of the theory, with different gauge symmetries (constraints) and different numbers of degrees of freedom (d.o.f.). The parity-even branch is nothing but standard GR with a cosmological constant which has become pure gauge under conformal invariance if matter is absent, or determined by matter (and so not an independent d.o.f.) if non-conformally invariant matter is present. In contrast, the parity-violating branch contains a genuinely new d.o.f. The work on the paper was split about equally between my co-author Professor Joao Magueijo and myself.

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