Summary of Professional Accomplishments

Waldemar Jarosław Kłobus

Faculty of Mathematics, Physics and Informatics
University of Gdańsk
Gdańsk
2023

Contents

1 Name

Waldemar Jarosław Kłobus

2 Diplomas, degrees conferred in specific areas of science or arts

- **Doctoral degree** in the field of physical sciences
  Institution: Faculty of Physics, Adam Mickiewicz University
  PhD thesis: *Properties of correlations in quantum mechanics and general probabilistic theories*
  Supervisor: Prof. Andrzej Grudka
  Assistant Supervisor: Dr. Karol Horodecki
  Revievers: Prof. Marek Kuś, Prof. Karol Życzkowski
  Doctoral thesis defense: 17.07.2014, Poznań
  Conferment of the academic degree: 19.09.2014, Poznań

- **Master's degree** in physics, specializing in theoretical physics
  Institution: Faculty of Physics, Adam Mickiewicz University
  MSc thesis: *Magnetic orderings, charge orderings, and phases separations in the extended Hubbard model*
  Supervisor: Prof. Stanisław Robaszkiewicz
  Grade: very good
  Conferment of the title: 7.07.2009, Poznań

3 Information on employment in research institutes or faculties / departments or school of arts

- 1.02.2021 – present: **Assistant professor** (permanent position)
  Institution: Institute of Theoretical Physics and Astrophysics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk
  Supervisor: Dr. Marek Krośnicki
  Research topics: multipartite correlations, multipartite quantum entanglement detection, methods of indirect entanglement detection

- 01.02.2018 – 31.01.2021: **Assistant professor** (post-doctoral position)
  Institution: Institute of Theoretical Physics and Astrophysics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk
  Funding source: The National Science Centre,
  research grant: Beethoven – 2016/23/G/ST2/04273
  Supervisor: Prof. Wiesław Laskowski
  Research topics: characterization of multipartite correlations, detection of multipartite quantum entanglement

- 12.2014 – 12.2016: **Research associate** (post-doctoral position)
  Institution: Faculty Physics, Adam Mickiewicz University
  Funding source: European Research Council,
  research grant: ERC Advanced Grant QOLAPS
  Supervisor: Prof. Andrzej Grudka
  Research topics: theoretical foundations and applications of quantum resources, quantum entanglement, nonlocality, and contextuality

- 04.2012 – 11.2014: **Doctoral research assistant**
  Institution: Faculty Physics, Adam Mickiewicz University
  Funding source: European Research Council,
  research grant: ERC Advanced Grant QOLAPS
  Supervisor: Prof. Andrzej Grudka
  Research topics: theoretical foundations and applications of quantum resources, quantum entanglement, nonlocality, and contextuality

4 Description of the achievements, set out in art. 219 para 1 point 2 of the Act

4.1 Title of the achievement

A cycle of thematically related scientific publications titled:

Analysis of complex systems and the influence of non-classical correlations on their properties

4.2 List of publications constituting the achievement

1) *'Transition from order to chaos in reduced quantum dynamics'*
W. Kłobus, P. Kurzyński, M. Kuś, W. Laskowski, R. Przybycień, K. Życzkowski
Physical Review E **105**, 034201 (2022)
DOI: `https://doi.org/10.1103/PhysRevE.105.034201`
ArXiv: `https://arxiv.org/abs/2111.13477`

2) *'Cooperation and dependencies in multipartite systems'*
W. Kłobus, M. Miller, M. Pandit, R. Ganardi, L. Knips, J. Dziewior, J. Meinecke, H. Weinfurter, W. Laskowski, T. Paterek
New Journal of Physics **23**, 063057 (2021)
DOI: `https://doi.org/10.1088/1367-2630/abfb89`
ArXiv: `https://arxiv.org/abs/2003.12489`

3) *'Gaussian state entanglement witnessing through lossy compression'*
W. Kłobus, P. Cieśliński, L. Knips, P. Kurzyński, W. Laskowski
Physical Review A **103**, 032412 (2021)
DOI: `https://doi.org/10.1103/PhysRevA.103.032412`
ArXiv: `https://arxiv.org/abs/2008.11733`

4) *'k-uniform mixed states'*
W. Kłobus, A. Burchardt, A. Kołodziejski, M. Pandit, T. Vertesi, K. Życzkowski, W. Laskowski
Physical Review A **100**, 032112 (2019)
DOI: `https://doi.org/10.1103/PhysRevA.100.032112`
ArXiv: `https://arxiv.org/abs/1906.01311`

5) *'Higher dimensional entanglement without correlations'*
W. Kłobus, W. Laskowski, T. Paterek, M. Wieśniak, H. Weinfurter
The European Physical Journal D **73**, 29 (2019)
DOI: `https://doi.org/10.1140/epjd/e2018-90446-6`
ArXiv: `https://arxiv.org/abs/1808.10201`

6) *'Measurement uncertainty from no-signaling and nonlocality'*
J. Łodyga, **W. Kłobus**, R. Ramanathan, A. Grudka, M. Horodecki, R. Horodecki
Physical Review A **96**, 012124 (2017)
DOI: `https://doi.org/10.1103/PhysRevA.96.012124`
ArXiv: `https://arxiv.org/abs/1702.00078`

7) *'Communication strength of boxes violating monogamy relations'*
W. Kłobus, M. Oszmaniec, R. Augusiak, A. Grudka
Foundations of Physics **46**, 620 (2016)
DOI: `https://doi.org/10.1007/s10701-015-9983-5`
ArXiv: `https://arxiv.org/abs/1408.1223`

4.3 Discussion of the motivation of the aforementioned works and the achieved results

The presented scientific achievement constitutes a series of thematically related research publications in collective works. A detailed discussion of the applicant's contribution to each of the works is presented in Chapter **1** of the attached document *List of scientific or artistic achievements which present a major contribution to the development of a specific discipline*, while the contributions of other co-authors are provided in separate statements attached as a separate document. The presentation has the following structure: first, the scientific objectives underlying the research on the specific topic will be outlined. Next, a brief overview of the context in which the research objectives were pursued, as well as the main results of the work, will be discussed. The content of the scientific publications constituting the series of works will be discussed sequentially in subsequent subsections. It should be noted that the adopted notation may feature repeated designations assigned to various concepts appearing in different subsections. This notation has been chosen to maintain consistency with the conventional notation used in published articles. Therefore, the meaning of the relevant expressions will be defined independently in each subsection (e.g., p, H, I, $\mathcal{D}$, C, etc.).

In the further part of this presentation, the following convention regarding references has been adopted:

- [H1]–[H7] refer to publications belonging to the presented scientific achievement,
- [O1]–[O17] refer to other publications co-authored by the applicant not included in the presented scientific achievement,
- [R1]–[R97] refer to other publications.

4.3.1 Introduction

The Nobel Prize in Physics awarded last year 'for experiments with entangled photons, establishing Bell inequality violation, and pioneering quantum information science' indicates the significant importance of this rapidly developing branch of physics that has emerged from considerations of fundamental questions regarding the structure of physical theories and their place in a broader context. From the very beginning of formulating the principles of quantum mechanics, its implications without classical counterparts were recognized. Considerations about uncertainty, resulting from the incompatibility of measurements, became the basis for inquiries into fundamental aspects of the theory itself. The EPR paradox [R1] was, in this respect, nothing but an attempt to understand the quantum-mechanical description of physical reality using particular correlations in a composite system. Three decades later, further considerations about the nature of correlations led to the formulation of a test [R2, R3] that allowed for experimental verification of intuitive assumptions regarding physical parameters describing reality.

These fundamentally philosophical, as it might seem, considerations eventually became the basis enabling the development of quantum information technologies, such as quantum cryptography [R4], quantum computing [R5], and secure algorithms [R6, R7], to name just a few. This clearly indicates the necessity of both fundamental research on quantum mechanics and its relationship with post-quantum principles, as well as its connections with classical effects, which can acquire new meanings in a quantum context. One of the ways that can contribute to further advancement of the field is the analysis of multipartite complex systems concerning the impact of their intrinsic correlations on their properties.

The motivation of the presented research in the context of this scientific achievement was to significantly broaden the understanding of the nature of multipartite correlations in complex systems and their resulting effects, both in terms of quantum correlations and their relationships with post-quantum constraints. The specific objectives focused on in the presented series of works can be summarized as follows:

- the research aimed to investigate the effects of breaking constraints on the correlations of multipartite systems,
- the research focused on examining non-quantum constraints on correlations in multipartite systems and their impact on the outcomes obtained within the quantum formalism,
- characterization of correlations in multipartite systems and investigating their utility in specific quantum protocols,
- presentation of a method for transferring quantum entanglement to systems with altered dimensionality,
- investigation of the characteristics of multipartite correlations and their utility for characterizing multipartite quantum entanglement,
- presentation of a method for constructing and investigating properties of multipartite maximally entangled states and their generalizations,
- investigating the influence of interactions in multipartite systems on their dynamics and determining conditions leading to chaotic behaviors.

4.3.2 Summary

In recent years, significant effort has been devoted to the analysis of non-signaling correlations, formulated as families of probability distributions $\{p(x, y, z|X, Y, Z)\}$ of measurement outcomes obtained in Bell experiments by spatially separated observers [R8, R9]. Non-signaling correlations are defined by probability distributions in which the probabilities of certain measurement outcomes obtained by specific observers (e.g., x, y) do not depend on the choices of measurement settings made by other observers (e.g., Z):

$$\sum_z p(x, y, z|X, Y, Z) = \sum_z p(x, y, z|X, Y, Z'). \tag{1}$$

In that case, the correlations of measurement outcomes X and Y satisfy $\langle XY \rangle_Z = \langle XY \rangle_{Z'} \equiv \langle XY \rangle$. One of the fundamental properties of multipartite non-signaling correlations is that they are monogamous [R10, R11, R12]. This property states that if in a system shared by, for example, three parties (Alice, Bob, and Eve), the correlations between two of them (Alice and Bob) violate a certain Bell inequality, then the correlations of the third party (Eve) with the others become significantly limited. This property has significant importance in applications in the field of cryptographic security [R13] based on the principle of non-signaling, as well as in the amplification of randomness [R14, R15], which are impossible with classical correlations. Recently, in the context of the problem of information loss in black holes, it has been pointed out [R16] that monogamy relations can be violated, so that by performing measurements on particles in a 'polygamous entangled state,' one can obtain correlations that violate (1) [R17]. Such signaling correlations can, in turn, be used for information transmission between users. It is worth mentioning that monogamy relations can be violated in quantum mechanics [R18, R19] when post-selection is allowed in Bell experiments.

In the paper [H1], we investigate the communication potential of a system as a result of violating monogamy relation. To do this, we consider a tripartite system and a specific monogamy relation defined for it [R11]. We demonstrate how, in the case of violation of the monogamy relation, it is possible to create a communication channel allowing for the transmission of classical information between users. We determine the minimum information capacity of such a channel depending on the degree of monogamy violation for the CHSH inequality and the general chain Bell inequality. Additionally, we present an alternative proof of the monogamy relation (for the CHSH inequality [R3] and the general chain Bell inequality [R20]), which provides insights into how the principle of non-signaling restricts the correlations obtained in Bell experiments.

The aforementioned non-signaling correlations have also been studied to understand the physical principles that influence the limitations of correlations obtained within the formalism of quantum mechanics and quantum effects (such as complexity reduction, quantum computational speedup, randomness amplification, or cryptographic key distribution) that could be achieved without relying on the principles of quantum mechanics. Indeed, to achieve certain non-classical effects, it is not necessary to apply the full quantum mechanical formalism. For example, secure cryptographic key distribution can be ensured solely based on the principles of non-signaling and Bell nonlocality [R13]. However, in this regard, the uncertainty principle [R21], as a non-classical effect embedded in the quantum formalism, has not been quantitatively derived solely based on the principles of non-signaling and Bell nonlocality.

The uncertainty principle can be characterized in two ways: as the uncertainty of preparation, stating that it is impossible to prepare a physical system in a state that simultaneously provides undisturbed statistics of two incompatible measurements [R22, R23], and as the uncertainty of measurement, stating that the act of measuring one observable disturbs the statistics of the measurement of another observable [R24, R25]. The uncertainty of measurement can be formulated in terms of a trade-off between the disturbance of the system's state and the amount of information gained about the state of the system [R26, R27].

In the work [H2], we obtain a quantitative description of measurement uncertainty in the form of a trade-off between the information that can be obtained in the measurement process of an observable and the disturbance of the system caused by the act of the measurement. We derive the uncertainty relation solely based on the principles of non-signaling and Bell nonlocality. To do this, we consider a scenario where two spatially separated users (Alice and Bob) possess a physical system exhibiting Bell nonlocality and perform consecutive measurements of observables on their subsystems. As shown, the act of measuring the first observable by Bob disturbs the statistics of the second measurement, even in the case of performing a so-called gentle measurement, where he does not gain complete knowledge of the outcome. We quantitatively demonstrate that the disturbance caused by the measurement depends not only on the amount of information obtained in the first measurement but also on the strength of mutual correlations characterized by the degree of violation of the appropriate Bell inequality. This indicates that measurement uncertainty in the case of multipartite systems can be regarded as an effect of the strength of mutual correlations, which is independent of the quantum formalism. The obtained trade-off between disturbance and obtained information may have potential applications in methods of cryptography based on the transmission of particles in a specific state (as in the BB84 protocol).

One of the fundamental problems in studying multipartite systems is the identification and measurability of different types of inter-subsystem correlations for their potential applications, both in classical and quantum systems. The issue of measuring correlations in complex systems finds applications in various scientific fields, such as genetics, neuroscience, sociology, economics, as well as in physics and information processing theory. Different measures of multipartite correlations have been analyzed, and they have been used in the study of diverse phenomena [R28, R29, R30, R31, R32, R33]. In physics and information processing, various measures of multipartite correlations have been subjected to analysis [R34, R35, R36]. Many of the quantitative measures of multipartite correlations require complex optimization processes that are challenging to implement numerically. As a result, finding computationally efficient methods for measuring these correlations is an ongoing research topic. The development of such efficient methods is essential for the practical application of these measures in real-world systems, allowing for a deeper understanding of the underlying correlations and their potential applications.

In the work [H3], we introduce a computationally accessible measure of inter-subsystem dependence in a multipartite system, defined as the informational gain obtained through cooperation among a group of parties sharing a multipartite system relative to another correlated system. This measure, referred to as dependence, is based on the utilization of conditional mutual information, a quantity widely used in classical and quantum information theory [R37, R38, R39, R40, R41, R42, R43, R44, R45]. We establish fundamental properties of the dependence measure and highlight its common features with measures of genuine multipartite correlations [R46]. Remarkably, the dependence measure characterizes different types of inter-subsystem relationships than quantum entanglement, although in specific cases, it can serve as a witness of quantum entanglement for mixed states. Furthermore, we demonstrate the potential application of the dependence measure by quantitatively determining the lower bound for rate of a quantum secret-sharing protocol [R47]. Additionally, we identify the optimal quantum states that maximize the dependence measure.

As shown above, quantum entanglement is qualitatively distinct from systems exhibiting non-zero multipartite dependence. Next, we investigate the relationship between quantum entanglement of a multipartite system ρ and its correlations characterized by the correlation tensor $T_{\mu_1 \dots \mu_N} = \mathrm{Tr}(\rho\, \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N})$, where the individual components will be referred to as elements. Pure N-partite states are entangled if and only if the sum of squares of all elements of the N-th order correlation tensor ($\mu_i \neq 0$ for all i) exceeds certain thresholds [R48, R49, R50, R51], which can be determined for any number of particles N and any dimension d of the subsystems. However, it turns out that for mixed N-qubit ($d = 2$) states, it is impossible to characterize entanglement using N-particle correlation tensor elements. There exist genuinely multipartite entangled N-qubit states with vanishing N-partite correlation tensor elements [R52, R53, R54]. Nevertheless, it is unclear whether this property is specific only to 2-dimensional systems.

In the paper [H4], we prove that the vanishing of N-partite correlation tensor elements is also possible for genuinely multipartite entangled states with higher dimensionality. By introducing a map that changes the sign of elements of specific order of the correlation tensor, we present a generalized construction scheme for N-qudit states with the property that all N-partite correlation tensor elements vanish. Furthermore, we provide an example of mixed states with vanishing N-partite correlation tensor elements that are genuinely multipartite entangled. Thus, we demonstrate that for mixed N-qudit states, there are no criteria for genuine multipartite entanglement based on N-partite correlation tensor elements.

Another topic related to multipartite quantum entanglement concerns the description of absolutely maximally entangled (AME) states. Pure AME states of N particles constitute a multipartite generalization of the concept of maximally entangled states [R55] of two particles, in the sense that each $\lfloor N/2 \rfloor$-partite reduced state leads to a maximally mixed state. Similarly, any state with the property that all k-partite reduced states are maximally mixed is referred to as a k-uniform state. It turns out that while for $N = 5$ and $N = 6$ qubits, one can find pure AME states [R56], in general, AME states do not exist for arbitrary N, as shown, for example, for $N = 4$ [R57] and $N = 7$ [R58]. However, the restriction on the existence of AME states for any number of particles N does not hold if the Hilbert space of the individual subsystems is sufficiently large [R59].

In the case where there is no pure k-uniform state for a given number of particles N, the question arises about the maximum purity that a k-uniform state of an N-particle system can have. The work [H5] is dedicated to analyzing this problem. Firstly, we characterize k-uniform states using the elements of the corresponding order tensor correlation, which allows for efficient numerical analysis of the problem. We present a general method for constructing k-uniform states using multipartite Pauli matrices (hereinafter referred to as generators) that satisfy certain conditions. We find explicit forms of generators that allow us to obtain all k-uniform states with the highest purity for $N \leq 6$ qubits (including pure AME states) and beyond. We show that the high purity of these k-uniform states, along with vanishing of lower-order elements of correlation tensor, enables exhibiting highly nonclassical properties, such as genuine multipartite entanglement and the potential to violate Bell inequalities.

In recent years, significant progress has been made in generating entangled states. However, in such experiments, questions arise concerning the confirmation of entanglement generation, detecting its presence, and quantitatively determining the degree of entanglement. Various methods of entanglement detection have been proposed, including Bell tests, measurement of entanglement witnesses, entropic inequalities, and others [R60, R61]. Quantum entanglement detection becomes more challenging for systems with higher dimensions and often requires quantum state tomography [R62], which involves performing measurements whose number grows exponentially with the system's dimensionality.

In the paper [H6], we address the problem of obtaining information about quantum entanglement within a high-dimensional system without the need for quantum state tomography. To achieve this, we introduce another system with a reduced dimensionality, which enables measurements to determine its state. We want to verify the entanglement of the first system, whose direct analysis is challenging. We show that with a specific type of interaction between the two systems, it is possible to transfer entanglement between them. This transfer allows for certifying the entanglement of the first system through the analysis of the second system with reduced dimensionality. Due to the dimensionality reduction of the second system, a complete transfer of the state from one system to the other is not possible. Nevertheless, we demonstrate that even with this lossy compression of the state, it is still possible to obtain information about the entanglement of the initial state.

One of the research topics covered in this presentation is the analysis of the dynamics of multi-particle systems leading to chaotic behavior. In recent years, considerable attention has been devoted to studying the properties of quantum counterparts of classical chaotic systems, aiming to better understand the connections between classical and quantum mechanics [R63]. In this context, several analyses have been conducted on the kicked rotator model as an example of classical chaotic dynamics and its corresponding unitary quantum evolution that occurs in a finite Hilbert space [R64, R65, R66, R67, R68, R69]. This model describes the behavior of a spin in a constant magnetic field subjected to a periodic sequence of nonlinear impulses (kicks). Recently, it has been shown [R70] that there is a close connection between the quantum entanglement entropy of a multi-particle system and the chaotic dynamics of the classical kicked rotator, which serves as a classical counterpart of an N-spin system. This suggests that quantum entanglement may play a fundamental role in formulating the quantum analogue of the Kolmogorov-Arnold-Moser theory.

In the considered scenarios, while the dynamics of the entire multi-particle system is unitary, the dynamics reduced to a selected subsystem can exhibit nonlinear behavior. In the work [H7], we analyze the properties of nonlinear dynamics of a single qubit obtained through the partial reduction of the rest of the system. In the model of interacting spin system, each particle is subjected to the action of amplitude damping channel. We show that depending on the damping parameter, the system exhibits diverse behaviors. The conducted analysis indicates that phenomena previously observed in classical systems, such as period-doubling and the onset of chaos in the Feigenbaum scenario [R71], also manifest in the reduced dynamics of quantum systems.

4.3.3 Communication potential of a multipartite system as a result of breaking monogamy relations

In the work [H1], we analyze correlations in a multipartite system that allow breaking the monogamy relation. In this case, the given system will also violate the principle of no-signaling. This raises a natural question of how such a signaling system can be used to transmit information between the parties sharing it and how much information can be transmitted in this way. To address this question, we first derive the monogamy relation in a characteristic way, distinct from those presented in the literature [R11], which helps to understand the constraints imposed by the no-signaling principle on the correlations obtained in the Bell experiment. Let us consider a physical system shared by three parties, each of whom performs measurements of a certain observable (X, Y, and Z respectively) yielding results ± 1 with a joint probability distribution $p(XYZ)$. The following inequality holds true

$$(-1)^i \langle XY \rangle_Z + (-1)^j \langle YZ \rangle_X + (-1)^k \langle XZ \rangle_Y \leq 1, \tag{2}$$

for any $i, j, k = 0, 1$ such that the sum $i + j + k$ is odd, while the lower index determines the context (measurement of the third observable) in which measurements of the other two observables were made. If we consider the CHSH scenario, where both Alice and Bob can measure one of two observables, and additionally, spatially separated Eve performs a measurement of another observable, for each context, there must exist a joint probability distribution $p(A_i B_j E)$, and therefore, the following inequalities must be satisfied

$$\begin{aligned}
\langle A_0 B_0 \rangle_E + \langle B_0 E \rangle_{A_0} - \langle A_0 E \rangle_{B_0} &\leq 1, &(3)\\
\langle A_1 B_0 \rangle_E + \langle B_0 E \rangle_{A_1} - \langle A_1 E \rangle_{B_0} &\leq 1, &(4)\\
\langle A_1 B_1 \rangle_E - \langle B_1 E \rangle_{A_1} + \langle A_1 E \rangle_{B_1} &\leq 1, &(5)\\
-\langle A_0 B_1 \rangle_E + \langle B_1 E \rangle_{A_0} + \langle A_0 E \rangle_{B_1} &\leq 1. &(6)
\end{aligned}$$

In the case of no-signaling correlations, we have that the average values of products (hereafter referred to as correlators) of the same observables do not depend on the contexts in which they are measured ($\langle XY \rangle_Z = \langle XY \rangle_{Z'}$). By summing up the above four inequalities, we straightforwardly obtain the monogamy relation for the CHSH inequalities of the form

$$CHSH_{AB} + 2\langle B_0 E \rangle \leq 4, \tag{7}$$

with $CHSH_{AB}$ as expected value of CHSH operator. The inequality serves as an upper bound on correlations obtained in measurements performed by three observers in any theories consistent with the no-signaling constraints. On the other hand, if a certain family of probability distributions (hereafter referred to as a box) violates the monogamy relation, then necessarily the no-signaling relation would also be violated, resulting in correlators measured in different contexts having different values ($\langle XY \rangle_Z \neq \langle XY \rangle_{Z'}$). This, in turn, allows for the transmission of information from the observer choosing between measurements of Z and Z' to the observers measuring correlations X and Y.

Let us assume that the monogamy relation (7) is violated by a certain value Δ. Then, by summing up the inequalities (3)–(6)

$$\langle A_0E\rangle_{B_0} - \langle A_0E\rangle_{B_1} + \langle A_1E\rangle_{B_0} - \langle A_1E\rangle_{B_1} + \langle B_1E\rangle_{A_1} - \langle B_1E\rangle_{A_0} \geq \Delta, \tag{8}$$

it turns out that for at least one pair of correlators (e.g., $\langle A_0E\rangle_{B_0}$ and $\langle A_0E\rangle_{B_1}$), their values differ by at least $\Delta/3$. In such a situation, users measuring the observables A_0 and E obtain different values of correlators depending on the context (the choice of measurement setting for B). Signaling the choice of measurement setting B_0/B_1 establishes an (asymmetric) binary information channel $\mathcal{C}_{B\to AE}$, whose capacity depends on the specific setup on which the measurements are performed. This scheme allows for quantitatively determining the communication strength of boxes that violate the monogamy relation based on the concept of classical channel capacities.

Note that the condition (8) does not uniquely determine which pair of correlators is responsible for breaking the monogamy relation, which practically means that signaling potentially can occur in other schemes than $B \to AE$. Furthermore, the box violating the monogamy relation may also exhibit signaling in a scheme from one or a pair of observers to another. For the sake of simplicity in further analysis, we focus on boxes with vanishing average values of individual observables, which exclude the possibility of such signaling schemes. It should be noted, however, that for each box violating the monogamy relation (7) by a specific value Δ, there exists a box with vanishing average values of individual observables that yields the same correlator values, thereby preserving the signaling structure. This allows for characterizing the box in terms of its communication strength.

To quantitatively determine the communication strength of the signaling box, we need to consider the possibility of signaling in all schemes resulting from (8), namely $B \to A_0E$, $B \to A_1E$, and $A \to B_1E$. We associate these schemes with (asymmetric) channels with capacities $C_{B\to A_0E}$, $C_{B\to A_1E}$, and $C_{A\to B_1E}$, respectively, which depend on the values of the associated correlators. The communication strength of the signaling box is defined as the minimum capacity of the optimal signaling scheme for the box violating the monogamy relation.:

$$C_\Delta = \min_{\mathcal{P}_\Delta} \max\{C_{B\to A_0E}, C_{B\to A_1E}, C_{A\to B_1E}\}, \tag{9}$$

where $\mathcal{P}_\Delta$ defines the space of all boxes that violate the monogamy relation with a given value Δ. In the paper, we demonstrate that the optimization over the space of signaling boxes $\mathcal{P}_\Delta$ can be replaced by the optimization over a polytope $\mathcal{Q}_\Delta$, which is determined by the inequalities

$$\begin{aligned}
\langle A_0E\rangle_{B_0} - \langle A_0E\rangle_{B_1} + \langle A_1E\rangle_{B_0} - \langle A_1E\rangle_{B_1} + \langle B_1E\rangle_{A_1} - \langle B_1E\rangle_{A_0} &\geq \Delta, &(10)\\
\langle A_0E\rangle_{B_0} + \langle A_0E\rangle_{B_1} + \langle A_1E\rangle_{B_0} - \langle A_1E\rangle_{B_1} - \langle B_1E\rangle_{A_1} - \langle B_1E\rangle_{A_0} &\geq \Delta, &(11)\\
\langle A_0E\rangle_{B_0} - \langle A_0E\rangle_{B_1} + \langle A_1E\rangle_{B_0} + \langle A_1E\rangle_{B_1} + \langle B_1E\rangle_{A_1} + \langle B_1E\rangle_{A_0} &\geq \Delta, &(12)\\
\langle A_0E\rangle_{B_0} + \langle A_0E\rangle_{B_1} + \langle A_1E\rangle_{B_0} + \langle A_1E\rangle_{B_1} - \langle B_1E\rangle_{A_1} + \langle B_1E\rangle_{A_0} &\geq \Delta, &(13)
\end{aligned}$$

resulting from (2) for specific sets of observables, and additionally, trivial conditions of the form $-1 \leq \langle XY\rangle_z \leq 1$.

Naturally, when the monogamy relation is satisfied, $\Delta = 0$ and thus $C_0 = 0$. The values of the communication strength of signaling boxes for $\Delta > 0$ obtained through numerical optimization are shown in Fig.1. For the case of maximum violation of the monogamy relation $\Delta = 2$, the optimization procedure can be done analytically, resulting in a value (rounded) of $C_2 = 0.158$. Furthermore, for any value of Δ, we can find a box (a family of probability distributions) that achieves the communicative power value determined by C_Δ:

$$p(A_iB_jE) = \frac{1}{4}\left[1 + A_iE\left(\frac{\Delta}{2}\delta_{j,0} + x\delta_{j,1}\right)\right](\delta_{ij,0}\delta_{A_iB_j,0} + \delta_{ij,1}\delta_{A_iB_j,-1}), \tag{14}$$

where $\delta_{m,n}$ is Kronecker delta, while the parameter x satisfies the equation

$$C((1+\Delta/2)/2, (1+x)/2) = C((1+x)/2, (1-x)/2), \tag{15}$$

where $C(q,r)$ is capacity of an (asymmetric) binary channel.

In the paper, we also present a generalization of the monogamy relation for the chained Bell inequality, where each of the two observers (Alice and Bob) can perform measurements on M dichotomic observables. Also in this case, with inequalities of the form

$$\langle A_{i+j}B_i\rangle_E - (-1)^j\langle B_iE\rangle_{A_{i+j}} + (-1)^j\langle A_{i+j}E\rangle_{B_i} \leq 1, \tag{16}$$

where $i = 1, \ldots, M-1$ and $j = 0, 1$ we obtain the monogamy relation for the chained Bell inequality [R20]

$$|I^M_{AB}| + 2|\langle B_0 E\rangle| \leq 2M. \tag{17}$$

Similarly as in the previous case, the violation of the monogamy relation implies the possibility of signaling in a scheme from one observer to a pair of observers, which means that the system has nonzero communication strength. Further analysis allows us to find a lower bound on the communication strength in the form

$$C^M_\Delta \geq 1 - H\big((1 + \tfrac{\Delta}{4M-2})/2\big), \tag{18}$$

(where H represents the value of binary entropy), which, for comparison, is also presented in Fig. 1.

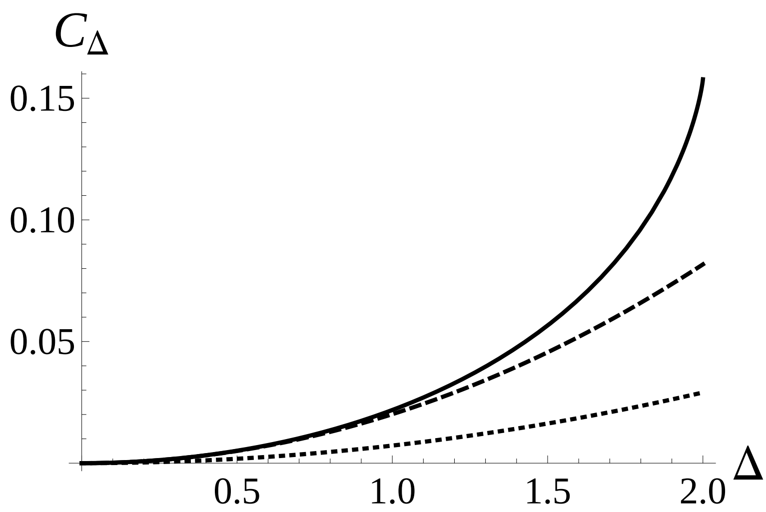

Figure 1: The communication strength C_Δ of the signaling box violating the monogamy relation (7) with value Δ (solid line). For comparison, the lower bound on the communication strength C^M_Δ of the signaling setup violating the monogamy relation for the chained Bell inequality with $M = 2$ (dashed line) and $M = 3$ (dotted line).

4.3.4 Disturbance vs information gain trade-off as a result of the no-signaling constraint and the strength of correlations in a multipartite system

In the work [H2], we derive the uncertainty principle expressed as a trade-off between disturbance and information gain based on two assumptions: the no-signaling principle and the violation of Bell inequality. We consider a scenario where two spatially separated users (Alice and Bob) have access to a physical system exhibiting Bell nonlocality, on which they perform consecutive measurements of observables on their respective subsystems. Notice that in the situation where Bob would measure not just one observable but two successive incompatible observables on his subsystem each time, the second measurement will be performed on a disturbed system. Therefore, in general, the statistics of Alice and Bob's outcomes do not guarantee the preservation of nonlocality of correlations.

To quantitatively determine the disturbance resulting from the measurement of one of the observables, we consider an equivalent measurement scenario. We assume that Bob's subsystem is correlated with an additional subsystem on which a third user (Charlie) performs a measurement of one observable corresponding to Bob's observable that he would initially measure gently. In general, we consider a three-party system where each party performs sharp measurements providing complete information about the variable values describing the system. Let us denote the probability distribution of outcomes for Allice and Bob by $p(a,b|A_x,B_y)$ and the probability distribution of outcomes for the three-party system by $\tilde{p}(a,b,b_1^g|A_x,B_y,B_1^g)$, where b_1^g represents the outcome of Charlie's measurement, which corresponds to the gentle measurement of observable B_1. Naturally, $\tilde{p}(a,b|A_x,B_y,B_1^g) = \sum_{b_1^g} \tilde{p}(a,b,b_1^g|A_x,B_y,B_1^g)$. In this scheme, we require Charlie's measurement not to disturb the statistics of Bob's outcomes conditioned on any measurement settings and Alice's measurement outcomes (no-signaling principle)

$$p(b_1|B_1,a,A_x) = \tilde{p}(b_1|B_1,B_1^g,a,A_x), \quad \text{for all } a,x. \tag{19}$$

Additionally, we require the outcome of Charlie's measurement to be correlated with the outcome of Bob's observable measurement B_1

$$\tilde{p}(b_1^g = i|b_1 = j, B_1, B_1^g, a, A_x) = \begin{cases} \frac{1}{2} + \epsilon & \text{for } i = j, \\ \frac{1}{2} - \epsilon & \text{for } i \neq j, \end{cases} \tag{20}$$

where ϵ determines the informational gain about the value of the variable B_1.

Now, we want to determine how much the gentle measurement of observable B_1 (in our case, performed as a sharp measurement of the correlated observable B_1^g) disturbs the state of the entire system. The natural measure of disturbance is the statistical distance between the distribution of outcomes obtained when measuring observable $B_y \neq B_1$ (i.e., before the gentle measurement) denoted as $p(b|B_y,a,A_x)$, and the distribution of outcomes obtained after this measurement denoted as $\tilde{p}(b|B_y,B_1^g,a,A_x)$. This distance is determined by

$$D_{a,x}(B_y) = \sum_b |p(b|B_y,a,A_x) - \tilde{p}(b|B_y,B_1^g,a,A_x)|. \tag{21}$$

In the paper, we consider the total disturbance $\mathcal{D}$ expressed as the values of disturbance in Bob's statistics averaged over all measurement settings and outcomes of Alice, given by

$$\mathcal{D} = \sum_{a,x} p(A_x)p(a|A_x) \sum_{y\neq 1} D_{a,x}(B_y). \tag{22}$$

As mentioned in the introduction, the value of disturbance caused by a measurement is related to the change in the nonlocal correlations. More precisely, we demonstrate that the following relation holds

$$n\mathcal{D} \geq |\beta(p) - \beta(\tilde{p})|, \tag{23}$$

where n determines the number of Alice's measurement settings, and the parameter β specifies the expectation value of the Bell operator calculated with the appropriate statistics p and $\tilde{p}$.

It should be noted that the selected Bell inequality generally do not encompass all observables that Bob can measure. In such a case, the above relation can become vacuous. To determine how much the measurement of observable B_1 can lead to disturbance concerning a specific Bell inequality, we introduce a quantity called significance $w(B_1)$. This is defined as the difference between the maximum value of the expected Bell operator for no-signaling boxes and the corresponding value in the case when the variable B_1 is deterministic

$$w = \beta^{\max} - \beta_1^{\max}, \tag{24}$$

and then we prove that the following monogamy relation holds true

$$\beta + w\langle B_1^g B_1 \rangle \leq \beta^{\max}. \tag{25}$$

Next, by expressing the value of information obtained as a result of the measurement by $\mathcal{I} = \langle B_1^g B_1 \rangle = 2\epsilon$, and the level of nonlocality by $\mathcal{L} = \beta^{\max} - \beta$, we obtain the uncertainty principle in the form of a trade-off between the amount of information gained from the measurement and the magnitude of the disturbance caused by the act of measurement

$$n\mathcal{D} \geq w\mathcal{I} - \mathcal{L}, \tag{26}$$

which constitutes the main result of the work.

It is worthwhile to apply the uncertainty relation to well-known measurement setups. First, we consider the CHSH measurement scheme, for which we relate the appropriate values characterizing the level of nonlocality $\mathcal{L}$. As a result, we obtain the uncertainty relation in the form of a linear constraint that provides a lower bound on the disturbance value in the following form

$$\mathcal{D} \geq 2\epsilon - \frac{1}{2}(4 - \beta_{\mathrm{CHSH}}). \tag{27}$$

The lower bound on the disturbance value is presented in Fig.2, emphasizing that this nontrivial bound is obtained independently of the formalism of quantum mechanics. We then compare the obtained result with values obtained within the framework of quantum mechanics. In this case, we assume that the sharp measurement of observable B_1 corresponds to the measurement defined by projection operators $\hat{P}_0$ and $\hat{P}_1$, while the gentle measurement is implemented by a measurement specified by Kraus operators $\hat{E}_0 = \sqrt{\frac{1}{2} + \epsilon}\hat{P}_0 + \sqrt{\frac{1}{2} - \epsilon}\hat{P}_1$ i $\hat{E}_1 = \sqrt{\frac{1}{2} - \epsilon}\hat{P}_0 + \sqrt{\frac{1}{2} + \epsilon}\hat{P}_1$. The statistics $\tilde{p}(b|b_1^g, B_y, B_1^g)$ are obtained from sharp measurements performed on the appropriate conditional states based on the outcomes obtained in the preceding gentle measurement. We demonstrate that the quantum formalism satisfies the conditions (19)–(20), thus corresponding to the presented earlier scheme, leading to the uncertainty relation. Interestingly, in the case of gentle measurement providing complete knowledge of the variable's value ($\epsilon = \frac{1}{2}$), we achieve exact agreement with the results obtained within the framework of quantum mechanics (Fig.2).

In the paper, we also consider a generalization of the above scheme concerning the chained Bell inequalities, where both Alice and Bob can make measurements in n different bases. In this case, the relation between disturbance and the information gain is expressed as:

$$\mathcal{D} \geq \frac{4}{n}\epsilon - \frac{1}{n}(2n - \beta_{\mathrm{chain}}), \tag{28}$$

which is presented in Fig.3.

We observe that the obtained uncertainty principle for non-signaling correlations in the CHSH scenario has the property that the lower bound on disturbance is trivial (i.e., equal to 0) as long as the information gain does not reach a specific threshold value $\epsilon_{th} = 0.293$. Considering a larger number of measurements (as in the case of chained Bell inequalities, where Alice and Bob can make measurements chosen from a larger set of bases) allows us to reduce the threshold value ϵ_{th} arbitrarily close to 0 (see Fig.3).

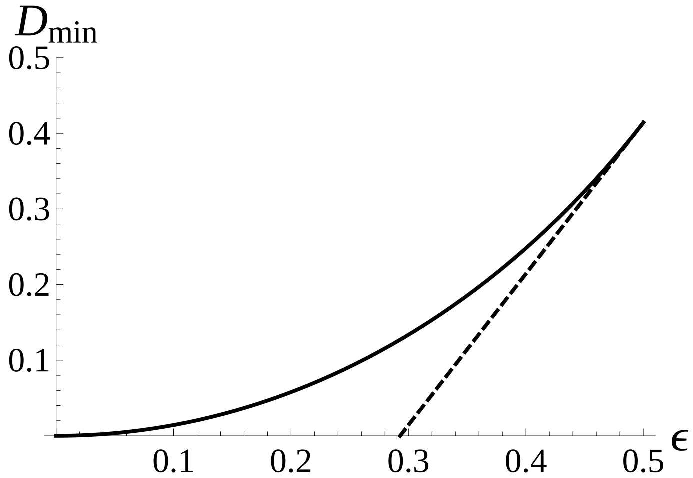

Figure 2: The lower bound on the disturbance value obtained from the non-signaling principle for the system violating the CHSH inequality up to the maximum quantum value $\beta_{\mathrm{CHSH}} = 2\sqrt{2}$ (dashed line). For comparison, the disturbance caused by the gentle measurement obtained within the framework of quantum mechanics is also presented (solid line).

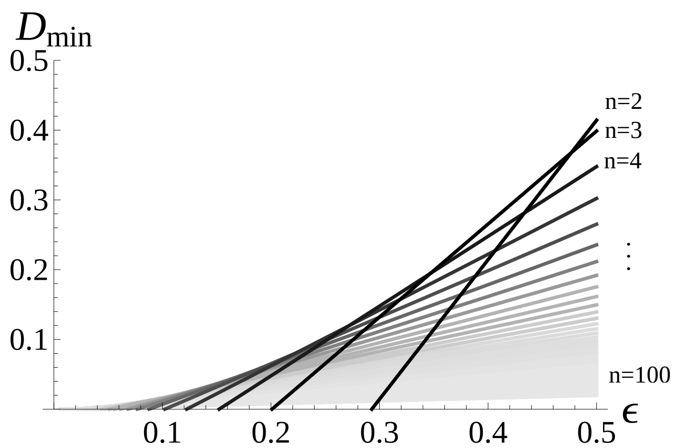

Figure 3: The lower bounds on the disturbance value obtained from the non-signaling principle for the system violating the chained Bell inequality are shown up to the maximum quantum value $\beta_{\mathrm{chain}} = 2n \cos \frac{\pi}{2n}$ for different values of possible measurement settings.

4.3.5 Cooperative information gain as a result of the interdependence in a multipartite system

In order to derive a measure of dependence, we consider a system shared by three parties (Alice, Bob, and Charlie), who make measurements of their respective observables (A, B and C). Variable A is statistically independent of the outcomes of variable B if the probability distribution of these variables satisfies $p(A|B) = p(A)$. Alternatively, statistical independence between two variables can be expressed in terms of Shannon entropy as $H(A|B) = H(A)$, and the statistical dependence between two variables can be represented by their mutual information $I(A : B) = H(A) - H(A|B)$. In the case of three variables, one of them can be independent of all others, e.g., $p(A|BC) = p(A)$, or conditionally independent of only one variable, e.g., $p(A|BC) = p(A|B)$. Thus, the statistical dependence between one variable and the others is expressed by the mutual information $I(A : BC)$, while the conditional dependence in the second case is expressed by the conditional mutual information $I(A : C|B)$. It is worth noting that the conditional mutual information $I(A : C|B) = I(A : BC) - I(A : B)$ expresses the informational gain through cooperation between Bob and Charlie, when they want to determine the value of Alice's variable. It is natural to define a quantity that, for three random variables with the joint probability distribution $p(ABC)$, represents the smallest possible informational gain resulting from cooperation between any two parties relative to the third one, given by

$$\mathcal{D}_3 = \min[I(A : C|B), I(A : B|C), I(C : B|A)]. \tag{29}$$

Such a quantity indeed expresses the interdependence between variables A, B, and C: the measure reaches a value of 0 if and only if there exists a variable for which there is a subset of the remaining parties possessing some knowledge about that variable, and this knowledge cannot be increased through cooperation with other parties.

To illustrate this idea, let's consider two probability distributions of three binary variables: $p(000) = p(111) = \frac{1}{2}$, and $p(000) = p(011) = p(101) = p(110) = \frac{1}{4}$. In both cases, the variables are strictly correlated. However, in the first case, knowing the value of the first variable automatically implies full knowledge of the value of the third variable, resulting in zero information gain from cooperation between the first and second parties. On the other hand, in the second case, each variable is independent of any other. Therefore, cooperation between two parties provides complete knowledge about the value of the third variable, resulting in a maximum value of 1 for the measure of interdependence.

In the case of complex systems with larger number of parties, we observe that changes occur not only in the possible cooperation patterns between subsets of participants (e.g., $I(X_1 : X_2X_3X_4) - I(X_1 : X_2X_3)$) but also in the number of variables whose values are sought through cooperation (e.g., $I(X_1X_2 : X_3X_4) - I(X_1X_2 : X_3)$). Therefore, quantifying the interdependence of a multipartite system requires optimizing over all possible cooperation patterns among users. Using the chain rule for mutual information, we demonstrate that the minimum value of the information gain for a given probability distribution is given by the mutual information of a pair of variables conditioned on the remaining ones. As a result, the interdependence of N variables is expressed as

$$\mathcal{D}_N = \min \; I(X_1 : X_2|X_3 \dots X_N), \tag{30}$$

where the minimum is taken over all permutations of N variables (subsystems). A non-zero value of dependence $\mathcal{D}_N$ for a given probability distribution indicates that any cooperation scheme among users leads to an increase in knowledge about the state of any other subset of variables.

For quantum systems of N particles in a state described by the density matrix ρ, the relative information is expressed in the form of von Neumann entropy, yielding

$$\mathcal{D}_N(\rho) = \min_{i,j}[S(\mathrm{Tr}_i\rho) + S(\mathrm{Tr}_j\rho) - S(\mathrm{Tr}_{ij}\rho) - S(\rho)], \tag{31}$$

where the minimum is taken over all possible choices of partial traces of sub-systems i and j. In this context, we notice that for three-particle quantum states, we have, for example, $I(X_1 : X_3|X_2) = S(X_1|X_2) + S(X_3|X_2) - S(X_1X_3|X_2)$, where for simplicity, we denote the state of the i-th subsystem as X_i. Knowing that the conditional entropy $S(X_1|X_2)$ characterizes the cost of merging quantum states X_1 with X_2 [R72], quantum conditional mutual information can be interpreted as an additional cost of merging states individually (X_1 with X_2 and X_3 with X_2) instead of merging all three states together (X_1X_3 with X_2). Consequently, the value of the interdependence $\mathcal{D}_N(\rho)$ determines the minimum additional cost of such a process.

In the work [H3], we determine the most important properties of the newly introduced quantity. The interdependence $\mathcal{D}_N$ reaches a minimum value of 0 for systems in which certain subsystems are independent of the rest, as well as for systems in which cooperation between any subsets of subsystems does not lead to an increase in knowledge about the states of the remaining subsystems, as mentioned earlier. For classical probability distributions, the value of $\mathcal{D}_N$ is upper-bounded by 1, which is achieved, for example, by a uniform distribution of three binary variables whose sum of values is even. For pure quantum states, the value of $\mathcal{D}_N(\rho)$ is also upper-bounded by 1, which is attained by the class of GHZ states. Additionally, we demonstrate that, for pure quantum states, the interdependence is determined by the minimum value of mutual information for the bipartite reduction of the N-partite pure state $|\Psi\rangle$

$$\mathcal{D}_N(|\Psi\rangle) = \min_{i,j}[S(\rho_i) + S(\rho_j) - S(\rho_{ij})]. \tag{32}$$

Unlike classical probability distributions of variables, the value of $\mathcal{D}_N(\rho)$ is upper-bounded by 2, which is achieved (for an even number of N qubits) for mixed states k-uniform states (with $k = N - 1$) of the form

$$\rho_{\max} = \frac{1}{2^N}\left(\sigma_0^{\otimes N} + (-1)^{N/2}\sum_{j=1}^{3}\sigma_j^{\otimes N}\right), \tag{33}$$

which serve as useful resources in multi-party communication protocols [R73]. We have demonstrated that k-uniform states of particles with a Hilbert space dimension of d and $k = N - 1$ (further discussed in [H5]) form a unique class of quantum states that achieve the upper bound of $\mathcal{D}_N(\rho)$.

We considered the interdependence measure $\mathcal{D}_N(\rho)$ as a potential measure of multipartite correlations. In the work [R46], three postulates were presented, which should be satisfied by measures or indicators of genuinely multipartite correlations (or genuinely multipartite entanglement). We have proved that the interdependence measure satisfies the first postulate, which states that if we add an uncorrelated particle to an N-particle system with $\mathcal{D}_N = 0$, then the resulting $(N+1)$-particle system has an interdependence measure of $\mathcal{D}_{N+1} = 0$. Next, we showed that the interdependence measure satisfies the second postulate, which states that if we divide one particle into two subsystems within an N-particle system with $\mathcal{D}_N = 0$, then the resulting $(N+1)$-particle system has an interdependence measure of $\mathcal{D}_{N+1} = 0$. On the other hand, the third postulate requires that a measure of genuinely multipartite correlations should not increase under local operations (although sometimes this condition is omitted in practice [R74, R75, R76]). However, the interdependence measure is not generally monotonic with respect to local operations. Nevertheless, we have shown that there exists an inequality that limits the increase in the value of interdependence after applying local operations

$$\overline{\mathcal{D}}_N \leq \mathcal{D}_N + I(X_1X_2 : X_3 \dots X_N) - I(X_1X_2 : \overline{X}_3 \dots \overline{X}_N), \tag{34}$$

where X_1 and X_2 represent the specific choice of sub-systems that minimize $\mathcal{D}_N$ (see the definition in (30)).

We note that the interdependence measure $\mathcal{D}_N$ characterizes correlations between particles of a different nature than genuinely multipartite entanglement or multipartite correlations determined by the N-partite correlation tensor elements. In the work [H4], we present a protocol through which one can create nontrivial N-particle states with vanishing N-particle correlation tensor elements from N-particle states possessing nonzero N-particle correlation tensor elements. Using this protocol, we demonstrate that a mixture of N-qubit Dicke states with 1 and $N-1$ excitations exhibits a nonzero value of interdependence $\mathcal{D}_N$ despite having vanishing N-partite correlation tensor elements. This illustrates that the N-particle correlation tensor elements are not responsible for the N-particle interdependence.

As mentioned earlier, multipartite systems in a classically correlated state exhibit $\mathcal{D}_N \leq 1$. It follows from the observation that systems with $\mathcal{D}_N > 1$ demonstrate the presence of quantum correlations. Indeed, from the definition of interdependence as the difference of two mutual informations $\mathcal{D}_N = I(X_1 : X_2X_3 \dots X_N) - I(X_1 : X_3 \dots X_N)$, we can see that since the second term is non-negative, the value of the first term is greater than one. This can be expressed using conditional entropy as

$$S_{X_1|X_2X_3\dots X_N}(\rho) < -1 + S(\rho_1). \tag{35}$$

For subsystems with equal dimensions we have $S(\rho_1) \leq 1$, hence the conditional entropy takes a negative value, which is a feature only exhibited by entangled states [R77]. However, the entangled state doesn't necessarily have to be genuinely multipartite entangled. This becomes particularly evident in the case of states that maximize the measure ρ_{max} (33), which are mixtures of correlated Bell states. We also consider the measure of interdependence for k-subsystems within N-partite states. In this context, if $\mathcal{D}_k > 1$, then every k-partite subsystem is entangled. An example of this is the 6-qubit absolutely maximally entangled (AME) state, for which all 4-particle subsystems are entangled, and $\mathcal{D}_4(\text{AME}(6,2)) = 2$.

We also investigate the opposite case, namely, whether there exist genuinely multipartite entangled states that exhibit $\mathcal{D}_N = 0$. We demonstrate that cluster states are an example of such states [R78]. Indeed, for pure cluster states, all reduced single-particle subsystems are maximally mixed, and there exists at least one pair of particles whose reduced state is also maximally mixed. Consequently, due to the relation (31), we obtain $\mathcal{D}_N = 0$ for genuinely multipartite entangled states. This means that the information about the state of one subsystem cannot be increased through cooperation with other subsystems, which also explains why these states cannot be utilized for tasks based on secret sharing [R79, R80]. Similar to the case of N-partite correlation tensor elements, we observe that the interdependence of subsystems quantified by the measure $\mathcal{D}_N$ is qualitatively different from genuinely multipartite entanglement.

Considering the above example, we notice that the interdependence measure $\mathcal{D}_N$ finds applications in determining the usefulness of resources for secret-sharing protocols. In our case, where the measure is defined as the minimum over permutations of parties, the resource can be utilized by any participant. In a prototypical example with three random variables described by the joint probability distribution $p(000) = p(011) = p(101) = p(110) = \frac{1}{4}$, the secret is the outcome of one user's variable, which is inaccessible to each of the other users individually. The interdependence measure $\mathcal{D}_N$ naturally determines the potential usefulness of the above box as a resource, and additionally, the secret can be generated by any of the users. In the context of secret sharing using quantum states [R47], we propose a protocol employing states ρ_{max} (33) that achieve the maximum value of the interdependence measure $\mathcal{D}_N$. Furthermore, for any quantum state, we derive an inequality that establishes the lower bound on rate of quantum secret sharing expressed by the value of the interdependence measure in the form of

$$\mathcal{R} \geq \mathcal{D}_N(\rho) - 1, \tag{36}$$

which indicates that for any systems with maximally mixed reduced states, the rate is positive. Additionally, we demonstrate that in the case of quantum systems with classically correlated reduced states (e.g., a system in the GHZ state), the minimum rate is strictly bounded by the value of $\mathcal{D}_N$.

4.3.6 Analysis of the relationship between quantum entanglement and correlation tensor

First, we consider the state of N qubits in the correlation tensor representation

$$\rho = \frac{1}{2^N} \sum_{\mu_1,\ldots,\mu_N=0}^{3} T_{\mu_1\ldots\mu_N}\, \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N}, \tag{37}$$

where the coefficients defining the tensor are given by its elements $T_{\mu_1\ldots\mu_N} = \mathrm{Tr}(\rho\, \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N})$. The construction of a state with vanishing N-partite correlation tensor elements presented in [R54] is based on applying a map that changes the sign of Pauli matrices $\mathcal{N} : \sigma_j \to -\sigma_j$ ($j = 1, 2, 3$) for all qubits in the initial state ρ. For a system consisting of an odd number of particles N, the density matrix after the mapping $\bar{\rho}$ satisfies $T_{j_1\ldots j_N}(\bar{\rho}) = -T_{j_1\ldots j_N}(\rho)$. In this case, the equiprobable mixture of these states $\rho_{\mathrm{nc}} = \frac{1}{2}(\rho + \bar{\rho})$ has vanishing N-partite correlation tensor elements for an odd number of particles.

In the case of subsystems with dimensions greater than 2, the state of N qudits in the correlation tensor representation can be expressed in the basis of operators $X^m Z^n$ ($m, n = 0, 1, \ldots, d-1$), where X and Z are the Heisenberg-Weyl operators. It turns out that a straightforward generalization of the $\mathcal{N}$ mapping for the qudit case, $\mathcal{N} : X^m Z^n \to \omega_d^m X^m Z^n$, $Z^n \to \omega_d^n Z^n$ with $\omega_d = \exp(2\pi i/d)$, is not a positive map in general. This prevents the construction of states with vanishing correlation tensor elements in a way analogous to the qubit case.

Alternatively, the state of N qudits in the correlation tensor representation can be expressed in the basis of Gell-Mann operators, which are a generalization of the Pauli operators for qudits

$$\begin{aligned} M^s_{j,k} &= \lambda_{j,k} + \lambda_{k,j} \quad \text{for } 1 \le j < k \le d, \\ M^a_{j,k} &= -i(\lambda_{j,k} - \lambda_{k,j}) \quad \text{for } 1 \le j < k \le d, \\ M^g_{j,k} &= \sqrt{\frac{2}{l(l+1)}} \left(\sum_{i=1}^{l} \lambda_{j,j} - l\lambda_{l+1,l+1} \right) \quad \text{for } 1 \le l \le d-1, \end{aligned} \tag{38}$$

with $\lambda_{j,k} = |j\rangle\langle k|$ in the standard basis. Numerating them sequentially, the state of N qudits in the correlation tensor representation can be written as follows

$$\rho = \frac{1}{d^N} \sum_{\mu_1,\ldots,\mu_N=0}^{d^2-1} T_{\mu_1\ldots\mu_N}\, M_{\mu_1} \otimes \cdots \otimes M_{\mu_N}. \tag{39}$$

We define the mapping of a d-dimensional quantum system as follows

$$\mathcal{N}_d(\cdot) := \sum_a \frac{M_a}{\sqrt{d-1}}\, (\cdot)^*\, \frac{M_a^\dagger}{\sqrt{d-1}}, \tag{40}$$

where the summation runs over all antisymmetric matrices $M^a_{j,k}$. It can be shown that the mapping of the Gell-Mann matrices yields

$$\mathcal{N}_d(M_0) = M_0, \tag{41}$$

$$\mathcal{N}_d(M_j) = -\frac{1}{d-1} M_j, \text{ for } j \neq 0, \tag{42}$$

and the mapping itself is a positive map, transforming any density matrix ρ of a qudit into a semi-positive definite matrix $\bar{\rho}$. Indeed, ρ^* does not change the eigenvalues of ρ, and the expression $\sum_a \frac{M_a}{\sqrt{d-1}}\, (\cdot)\, \frac{M_a^\dagger}{\sqrt{d-1}}$ represents a POVM map with Kraus operators given by $K_a = M_a/\sqrt{d-1}$.

In the case of a composite N-qudit system, applying the mapping (40) to each qudit results in the state

$$\bar{\rho} = (\mathcal{N}_d \otimes \cdots \otimes \mathcal{N}_d)(\rho), \tag{43}$$

the correlation tensor coefficients of which $\bar{T}_{j_1\ldots j_N}$ are related to the correlation tensor coefficients of the initial state (39) through

$$\bar{T}_{j_1\ldots j_N} = \frac{(-1)^N}{(d-1)^N} T_{j_1\ldots j_N}. \tag{44}$$

Therefore, we see that for an odd number of particles N, the state $\bar{\rho}$ has rescaled N-partite correlation tensor elements with a changed sign compared to the state ρ. This allows the construction of a state in the form of an imbalanced mixture

$$\rho_{\text{nc}} = p\,\rho + (1-p)\,\bar{\rho}, \tag{45}$$

with $p = \frac{1}{1+(d-1)^N}$ having the property that all N-partite correlation tensor elements of the state ρ_{nc} vanish.

To illustrate the construction of states with vanishing N-particle correlation tensor elements, we can consider the case of a state of a system consisting of three qutrits

$$\begin{aligned}\rho \;&=\; \frac{1}{27} M_0 \otimes M_0 \otimes M_0 + \frac{1}{18} \sum_{\pi_{i00}} \sum_{i=1}^{8} T_{i00}\; M_i \otimes M_0 \otimes M_0 \\ &+ \frac{1}{12} \sum_{\pi_{ij0}} \sum_{i,j=1}^{8} T_{ij0}\; M_i \otimes M_j \otimes M_0 + \frac{1}{8} \sum_{i,j,k=1}^{8} T_{ijk}\; M_i \otimes M_j \otimes M_k,\end{aligned} \tag{46}$$

where π_{ijk} denotes permutations of the indices ijk. After applying the mapping $\mathcal{N}$, the state ρ takes the form

$$\begin{aligned}\rho \;&=\; \frac{1}{27} M_0 \otimes M_0 \otimes M_0 + \frac{1}{18} \sum_{\pi_{i00}} \sum_{i=1}^{8} T_{i00}\; M_i \otimes M_0 \otimes M_0 \\ &+ \frac{1}{12} \sum_{\pi_{ij0}} \sum_{i,j=1}^{8} T_{ij0}\; M_i \otimes M_j \otimes M_0 + \frac{1}{8} \sum_{i,j,k=1}^{8} T_{ijk}\; M_i \otimes M_j \otimes M_k,\end{aligned} \tag{47}$$

while the mixture of states (45) gives a state in the form

$$\rho_{\text{nc}} = \frac{1}{27} M_0 \otimes M_0 \otimes M_0 + \frac{1}{18} \sum_{\pi_{i00}} \sum_{i=1}^{8} T'_{i00}\; M_i \otimes M_0 \otimes M_0 + \frac{1}{12} \sum_{\pi_{ij0}} \sum_{i,j=1}^{8} T'_{ij0}\; M_i \otimes M_j \otimes M_0, \tag{48}$$

whose correlation tensor elements are determined by $T'_{i00} = -\frac{1}{3} T_{i00}(\rho)$ and $T'_{ij0} = \frac{1}{3} T_{ij0}(\rho)$ (similarly for the other permutations of indices).

Next, we demonstrate that there exist genuinely multipartite entangled states whose N-particle correlation tensor elements vanish. For this purpose, we consider a state of three qutrits, which can be obtained using the construction shown above, starting from an initial state given by $\rho = |D^1_{3,3}\rangle\langle D^1_{3,3}|$, where

$$|D^1_{3,3}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{49}$$

The construction of the state $\rho_{\text{nc}}(D^1_{3,3})$ ensures the vanishing of the 3-particle correlation tensor elements. On the other hand, by calculating the entanglement monotone W introduced in the paper [R81] for the considered state, we find that $W(\rho_{\text{nc}}) = 0.0444$, indicating the presence of genuinely multipartite entanglement. We have thus shown that genuinely multipartite entanglement of a N-qutrit system cannot be characterized solely based on the N-partite correlation tensor elements.

In the next work [H5], we analyze the problem of finding k-uniform states with the highest purity. Firstly, we observe that if we define the correlation length r of the subsystems of a system in the state ρ as

$$M_r(\rho) = \sum_{\pi} \sum_{i_1,i_2,\ldots,i_r=1}^{3} T^2_{\pi(i_1 i_2 \ldots i_r)}, \tag{50}$$

then the property of k-uniformity for an N-partite state can be expressed in terms of the elements of the correlation tensor as follows

$$M_r(\rho_N^k) = 0. \tag{51}$$

Given the relation between the total correlation length of a state and its purity

$$\sum_{r=1}^{N} M_r(\rho) = 2^N \mathrm{Tr}\rho^2 - 1, \tag{52}$$

we observe that for a given purity of the state, the property of k-uniformity implies a concentration of correlations that occur exclusively among a larger number of particles ($r > k$) in the subsystems. Consequently, one can expect that k-uniform states with high purity will be characterized by strong nonclassical properties, such as genuine multipartite entanglement and Bell nonlocality. Indeed, as we demonstrated in the further part of the paper, all the found k-uniform states with a purity of at least $1/2$ are genuinely multipartite entangled and also exhibit a high probability of violating local realism bounds [R82, R83].

Next, we present a scheme for constructing k-uniform states based on N-qubit Pauli matrices in the form

$$G = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}, \tag{53}$$

where $i \in \{0, x, y, z\}$. Let us consider a set of N-qubit operators (hereafter referred to as generators) given by

$$\mathcal{G} = \{G_1, \ldots, G_m\}, \tag{54}$$

satisfying the following properties: 1) commutation, $[G_i, G_j] = 0$ for all i, j; 2) independence, $G_1^{i_1} \ldots G_m^{i_m} \propto \mathbb{1}$ only for $i_1 = \cdots = i_m = 0$ with $i_j = \{0,1\}$; and 3) k-uniformity, i.e., the product of generators $G_1^{i_1} \ldots G_m^{i_m}$ ($i_j = \{0,1\}$) results in a product of 1-qubit Pauli matrices (53) with at most $N-k-1$ matrices σ_0. Then the normalized sum of products of generators

$$\rho = \frac{1}{2^N} \sum_{j_1,\ldots j_m=0}^{1} G_1^{j_1} \ldots G_m^{j_m}, \tag{55}$$

represents the density matrix of a k-uniform state. We show the physical validity of the state (55) by demonstrating that

$$\rho = \frac{1}{2^N} (\mathbb{1} + G_1)(\mathbb{1} + G_2) \ldots (\mathbb{1} + G_m), \tag{56}$$

which allows us to show that the eigenvalues of the matrix (55) are 0, or 2^{m-N}. Comparing (55) with (37), we notice that the constructed state ρ has 2^m tensor correlation elements with values of ± 1. Due to (52), this also allows us to determine the purity of the state ρ as

$$\mathrm{Tr}\rho^2 = \frac{1}{2^N} 2^m = 2^{m-N}. \tag{57}$$

Therefore, we see that the purity of the state directly depends on the cardinality of the set of generators $\mathcal{G}$. The problem of constructing a k-uniform state boils down to finding the largest set of generators $\mathcal{G}$. Naturally, in the case of k-uniform states with $k = \lfloor N/2 \rfloor$, the number of generators equal to $m = N$ leads to the construction of AME states with purity 1.

Regardless of the previously presented construction of k-uniform states, in special cases, it is possible to construct a k-uniform state when a $(k-1)$-uniform state with k odd is given. We observe that by using the mapping introduced earlier in Eq. (43) [H4], it is possible to obtain a state with vanishing k-particle tensor correlation elements

$$\rho_N^k = \frac{1}{2}(\rho_N^{k-1} + \bar{\rho}_N^{k-1}), \tag{58}$$

as a result, the obtained state will be a k-uniform state. However, in this case, the purity of the obtained state is reduced by a factor of 2 (in the case of an N-qubit system) compared to the purity of the initial $(k-1)$-uniform state.

In the following, we present exemplary sets of generators $\mathcal{G}$ which uniquely determine the corresponding k-uniform states with the highest purity for different values of the number of particles N and the degree of uniformity k.

- N-partite 1-uniform states obtained from $m = N$ generators

  $$G_1 = ZX\cdots XX, \quad G_2 = XZ\cdots XX, \quad \cdots, \quad G_{N-1} = XX\cdots ZX, \quad G_N = XX\cdots XZ, \tag{59}$$

  which are equivalent with N-partite GHZ states.

- N-partite $(N-1)$-uniform states obtained from $m = 1$ generator (for even N)

  $$G_1 = Z\cdots Z, \tag{60}$$

  and from $m = 2$ generators (for odd N)

  $$G_1 = X\cdots X, \quad G_2 = Z\cdots Z, \tag{61}$$

  which in this case allows for obtaining the generalized Smolin entangled state [R84] in the form of (33), achieving the upper bound of N-partite dependence measure $\mathcal{D}_N$ [H3].

- 4-partite 2-uniform state obtained from $m = 3$ generators

  $$G_1 = XXXX, \quad G_2 = YYYY, \quad G_3 = \mathbb{1}XYZ, \tag{62}$$

  form an equiprobable mixture of 2 pure states, as there is no pure AME state for $N = 4$.

- 5-partite 2-uniform states obtained from $m = 5$ generators

  $$G_1 = \mathbb{1}XYXY, \quad G_2 = \mathbb{1}ZXX\mathbb{1}, \quad G_3 = XYY\mathbb{1}Z, \quad G_4 = XZYZY, \quad G_5 = ZXZ\mathbb{1}X, \tag{63}$$

  are equivalent to pure 5-qubit AME states obtained in connection with quantum error correction codes [R85].

- 5-partite 3-uniform states obtained from $m = 4$ generators

  $$G_1 = \mathbb{1}XXXX, \quad G_2 = \mathbb{1}YYYY, \quad G_3 = X\mathbb{1}XYZ, \quad G_4 = Y\mathbb{1}YZX, \tag{64}$$

  which are genuinely 5-partite entangled states with only non-vanishing 4-particle tensor correlation elements.

- 6-partite 3-uniform states obtained from $m = 6$ generators

  $$\begin{aligned} &G_1 = \mathbb{1}\mathbb{1}ZZZZ, \quad G_2 = \mathbb{1}XYZ\mathbb{1}X, \quad G_3 = \mathbb{1}ZXY\mathbb{1}Z, \\ &G_4 = XYZ\mathbb{1}Z\mathbb{1}, \quad G_5 = Z\mathbb{1}Z\mathbb{1}XY, \quad G_6 = ZYYZZY, \end{aligned} \tag{65}$$

  corresponding to pure 6-qubit AME states [R86].

- 7-partite 3-uniform states obtained from $m = 6$ generators

$$G_1 = Y\mathbb{1}YXZXZ, \quad G_2 = \mathbb{1}XXYYZZ, \quad G_3 = ZXYYXZ\mathbb{1},$$
$$G_4 = ZZ\mathbb{1}YXXY, \quad G_5 = YY\mathbb{1}Y\mathbb{1}\mathbb{1}Y, \quad G_6 = ZXYZ\mathbb{1}YX, \tag{66}$$

form an equiprobable mixture of 2 pure states, as for $N = 7$, there is no pure AME state [R58].

For the presented above k-uniform states, as well as for all other k-uniform states presented in the work [H5], we have numerically confirmed that they exhibit the highest purity for a given number of particles N and homogeneity degree k. The maximum purity of a specific k-uniform state determines the cardinality of the set of generators $\mathcal{G}$ that satisfy the previously mentioned conditions 1)–3). While a rigorous characterization of the maximum number of generators defining the set $\mathcal{G}$ for given N and k remains an open problem, in certain cases, we point out a connection between the generators in $\mathcal{G}$ and specific rows of orthogonal arrays $OA(r, N, 4, s)$ [R87]. We demonstrate that for a given orthogonal array $OA(r, N, 4, s)$, there exists an assignment of individual rows to N-qubit Pauli matrices, enabling the construction of $(N - s)$-uniform states, as long as these matrices satisfy the commutation 1) and mutual independence 2) conditions. For example, the orthogonal array $OA(16, 4, 4, 2)$ allows us to find $m = 3$ generators $G_1 = \mathbb{1}YYY, G_2 = XZYX, G_3 = YXZY$, enabling the construction of a 4-qubit 2-uniform state. In the same vein, the orthogonal array $OA(16, 5, 4, 2)$ allows us to find generators (specified in equation (64)) for a 5-qubit 3-uniform state, the orthogonal array $OA(64, 6, 4, 3)$ allows us to find generators (specified in equation (65)) for a 6-qubit 3-uniform state, and similarly $OA(32, 9, 4, 2)$ for a 9-qubit 5-uniform state, and $OA(4096, 12, 4, 5)$ for a 12-qubit 5-uniform state.

The construction scheme of k-uniform states utilizing the concept of generators defined by conditions 1)–3) can be easily generalized to the case of systems with particles described by a higher-dimensional Hilbert space. In this case, we consider operators $G_i^{(d)}$, which are N-qudit Heisenberg-Weyl matrices $S_{kl}^{(d)} = (X^{(d)})^k (Z^{(d)})^l$, allowing the construction of N-qudit k-uniform states

$$\rho = \frac{1}{d^N} \sum_{j_1,\ldots,j_m=0}^{d-1} (G_1^{(d)})^{j_1} \ldots (G_m^{(d)})^{j_m}, \tag{67}$$

whose purity is d^{m-N}. The above construction allows obtaining, among others:

- N-particle 1-uniform states obtained from $m = N$ generators

$$G_1^{(d)} = Z^{(d)} X^{(d)} \cdots X^{(d)} X^{(d)}, \quad G_2^{(d)} = X^{(d)} Z^{(d)} \cdots X^{(d)} X^{(d)},$$
$$\cdots,$$
$$G_{N-1}^{(d)} = X^{(d)} X^{(d)} \cdots Z^{(d)} X^{(d)}, \quad G_N^{(d)} = X^{(d)} X^{(d)} \cdots X^{(d)} Z^{(d)}, \tag{68}$$

which are equivalent with N-partite GHZ states;

- 4-particle 2-uniform states obtained from $m = 4$ generators

$$G_1^{(3)} = \mathbb{1}^{(3)} Z^{(3)} Z^{(3)} (Z^{(3)})^2, \quad G_2^{(3)} = \mathbb{1}^{(3)} X^{(3)} X^{(3)} (X^{(3)})^2,$$
$$G_3^{(3)} = Z^{(3)} \mathbb{1}^{(3)} Z^{(3)} Z^{(3)}, \quad G_4^{(3)} = X^{(3)} \mathbb{1}^{(3)} X^{(3)} X^{(3)}, \tag{69}$$

which are pure AME states of 4 qutrits (in contrast to 4 qubits, for which a pure AME state does not exist).

It should be noted that in cases where a pure k-uniform state does not exist, the maximum purity achieved by an N-qudit k-uniform state is $1/d$.

4.3.7 Analysis of quantum entanglement transfer to a system with reduced dimensionality

The work [H6] is dedicated to a method for transferring quantum entanglement from one system to another with a different dimensionality, allowing for an effective determination of entanglement without directly analyzing the entangled system. In the first step, we consider a two-subsystem operation called controlled rotation (CROT), which serves as the fundamental element of the transfer protocol. We define a unitary operation on a two-particle qudit-qubit system, where the qubit undergoes a rotation around the y-axis, controlled by the state of a d-dimensional system, given by

$$U_{\mathrm{CROT}} = \sum_{j=0}^{d-1} |j\rangle\langle j| \otimes \exp\left(-i\sigma_y \xi_j\right), \tag{70}$$

where the parameter ξ_j depends on the state of the qudit, and $\xi = \frac{j\pi}{2(d-1)}$. As we demonstrate, this defined operation allows for the transfer of information about the state of d-dimensional qudit, which can be read out by performing a suitable measurement on the 2-dimensional system. As an example, we can choose a qudit whose state is characterized by a single parameter p, given by

$$|\psi(p)\rangle_A = \sum_{k=0}^{d-1} \sqrt{\binom{d-1}{k} p^k (1-p)^{d-1-k}} |k\rangle. \tag{71}$$

After attaching a qubit in the state $|0\rangle$ and performing the CROT operation on the qudit-qubit system, it becomes possible to determine the value of the parameter p, which characterizes the state of the qudit, by a simple measurement of the observable σ_z on the qubit. We demonstrate that, in particular, for $d \to \infty$, the value of the parameter p is determined by $p = (1/\pi) \arccos \mathrm{Tr}(\rho_B \sigma_z)$.

Next, we analyze the application of the CROT operation for entanglement transfer between systems of different dimensions. Let us assume that we have a system A composed of a pair of d-dimensional qudits in the state $|\psi\rangle_A = \sum_{j,l=0}^{d-1} a_{jl} |j\, l\rangle$. After attaching an additional system B consisting of two qubits in the state $|0\rangle$ and performing the CROT operation on the appropriate qudit-qubit pairs, we obtain the state

$$\begin{aligned} |\Psi\rangle_{AB} &= U_{CROT}^{\otimes 2}(|\psi\rangle_A \otimes |00\rangle_B) \\ &= \sum_{j,l=0}^{d-1} a_{jl}\ |jl\rangle_A \otimes \Big[\cos\xi_j \cos\xi_l |00\rangle_B \\ &\quad + \cos\xi_j \sin\xi_l |01\rangle_B + \sin\xi_j \cos\xi_l |10\rangle_B + \sin\xi_j \sin\xi_l |11\rangle_B\Big], \end{aligned} \tag{72}$$

which in principle can be strongly four-partite entangled, which means that its individual subsystems (A and B) may be in a separable state. To achieve entanglement transfer from subsystem A to B, we perform a local projection of subsystem A onto the state $|++\rangle_A$, where $|+\rangle = 1/\sqrt{d} \sum_{k=0}^{d-1} |k\rangle$, and as a result we obtain the composite state of the system AB in the form

$$\begin{aligned} |\tilde{\Psi}\rangle_{AB} &\propto |++\rangle_A \sum_{j,l=0}^{d-1} a_{jl} \Big[\cos\xi_j \cos\xi_l |00\rangle_B \\ &\quad + \cos\xi_j \sin\xi_l |01\rangle_B + \sin\xi_j \cos\xi_l |10\rangle_B + \sin\xi_j \sin\xi_l |11\rangle_B\Big]. \end{aligned} \tag{73}$$

We observe that in the case where the first subsystem A was initially in a maximally entangled state defined by $a_{jl} = \delta_{jl}/\sqrt{d}$, the resulting system B after the above-described transfer scheme will be in a state whose overlap with the maximally entangled state of two qubits $\frac{1}{\sqrt{2}}(|00\rangle +$

$|11\rangle$) decreases with dimension d, but asymptotically ($d \to \infty$) reaches the value of $\pi^2/(\pi^2 + 4) \approx 0.712$. In the special case where the subsystem A consists of a qubit pair ($d = 2$), we demonstrate that the transfer operation using CROT operation leads to an exact transfer of the state from subsystem A to subsystem B.

Next, we investigate the entanglement transfer protocol when the system A is in a mixed state. Similar to the case when the subsystem A is in a pure state, we show that for $d = 2$, the transfer operation leads to an exact transfer of the state of subsystem A, ρ_A, to subsystem B. Additionally, we prove (for any dimensionality of subsystem A) that if the system A consisting of two qudits is in a separable state, the operation of entanglement transfer to the two-qubit system B will also remain separable. Consequently, we conclude that the entanglement transfer protocol utilizing the CROT operation allows subsystem B to serve as a entanglement witness for subsystem A with a different dimensionality.

We also consider the quantum entanglement transfer protocol in the scenario where the system A has a continuous spectrum. In this case, we define the CROT operation as

$$U_{\mathrm{CROT}} = \int dx |x\rangle\langle x| \otimes (\cos x \mathbb{1} - i \sin x \sigma_y). \tag{74}$$

When considering the action of the CROT operation on a system A in a Gaussian state characterized by two parameters (σ, m)

$$|\psi(\sigma, m)\rangle_A = \int dx \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(x-m)^2}{4\sigma^2}} |x\rangle, \tag{75}$$

to which we attach a qubit in the state $|0\rangle$ we observe that after performing the CROT operation on the system $|\psi(\sigma, m)\rangle_A|0\rangle_B$, the qubit state tomography allows us to find the parameters describing the initial state of the system A, namely $\sigma^2 = -(1/4)\ln||\vec{b}||^2$ and $m = \operatorname{arc}\cot(b_z/b_x)/2$, with $\vec{b}$ as a Bloch vector defining the state of the qubit subsystem B.

Next, we proceed with the analysis of entanglement transfer for the system A in a Gaussian state [R88, R89]

$$|\psi(\sigma, \Sigma)\rangle_A = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \frac{1}{(2\pi\sigma\Sigma)^{1/2}} e^{-\frac{(x_1+x_2)^2}{8\sigma^2}} e^{-\frac{(x_1-x_2)^2}{8\Sigma^2}} |x_1\rangle|x_2\rangle, \tag{76}$$

whose entanglement is measured by the purity of the subsystem [R90], given by $P = 2\sigma\Sigma/(\sigma^2 + \Sigma^2)$, and it is entangled for $\sigma \neq \Sigma$. We demonstrate that the entanglement transfer defined similarly to the previous case for states with a discrete spectrum, based on the projection of subsystem A onto the state

$$|x_1^+ x_2^+(\Gamma)\rangle = \int dx_1 \int dx_2 \frac{1}{(2\pi\Gamma^2)^{1/2}} e^{-\frac{(x_1^2+x_2^2)}{4\Gamma^2}} |x_1\rangle|x_2\rangle, \tag{77}$$

produces the target state of the two qubits in the form

$$|\tilde{\Psi}\rangle_B \propto a_+|00\rangle + a_-|11\rangle, \tag{78}$$

with

$$a_\pm = \frac{e^{-\frac{2\Sigma^2\Gamma^2}{\Sigma^2+\Gamma^2}} \pm e^{-\frac{2\sigma^2\Gamma^2}{\sigma^2+\Gamma^2}}}{\sqrt{\frac{(\sigma^2+\Gamma^2)(\Sigma^2+\Gamma^2)}{\sigma\Sigma\Gamma^2}}}, \tag{79}$$

where the highest probability of a successful projection is achieved for $\Gamma = \sqrt{\sigma\Sigma}$ and equals $4\sigma\Sigma/(\sigma + \Sigma)^2$. We notice that here we are dealing with a trade-off: the higher the level of entanglement of the output system A (the lower the purity of its subsystem), the lower the probability of successful projection required in the transfer protocol.

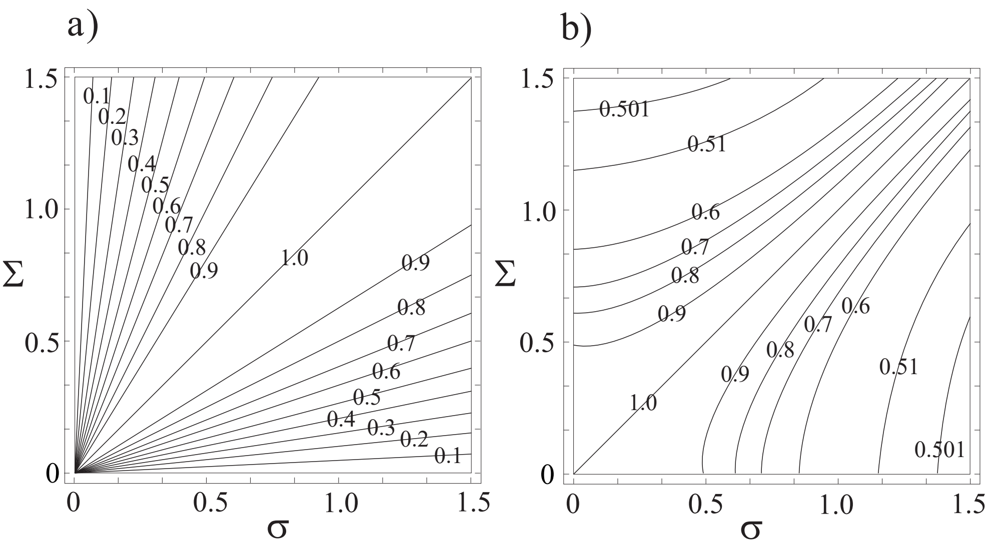

Figure 4: The purity of the subsystem as a function of the parameters σ and Σ is shown for (a) the output system A and (b) the system B after the entanglement transfer.

Considering that the analysis of system A is performed indirectly by examining system B, the obtained result can be compared with the direct method of determining the entanglement of system A by analyzing the purity of its subsystem. Figure 4 presents the purity of the subsystems of each system, namely the initial system A and the two-qubit system B in the state $|\tilde{\Psi}\rangle_B$ (78) after the entanglement transfer. It should be noted that a necessary and sufficient condition for entanglement of the Gaussian state of system A based on methods determined in [R91, R92], is $(\sigma^2 - \Sigma^2)^2 > 0$, which is consistent with the conclusions based on the analysis of the state of the two-qubit system B obtained during the entanglement transfer process.

4.3.8 Chaotic dynamics as a result of quantum interactions in a complex system

In the work [H7], we analyze the behavior of an interacting system of N particles, demonstrating that the effective dynamics of a single qubit resulting from mutual interactions exhibits nonlinear characteristics, leading to chaotic behaviors. We consider a system of N interacting qubits initially in a product state, where the interactions are governed by a Hamiltonian operator of the form

$$H = \frac{g}{2(N-1)} \left(\sum_{n=1}^{N} \sigma_z^{(n)} \right)^2 . \tag{80}$$

In the subsequent analysis, we focus on the effective dynamics of a single qubit ($n = 1$), considering its initial state to be $\rho = \sum_{j,k=1}^{d} \rho_{j,k} |j\rangle\langle k|$. We observe that the Hamiltonian operator can be separated into two commuting parts

$$H = H_1 + H_{\text{env}}, \tag{81}$$

where H_1 is the operator acting in the space of the selected qubit, and H_{env} is the operator acting in the space of the remaining $N-1$ qubits. Therefore, the dynamics of the selected qubit in the system is determined by

$$e^{-iH_1 t} = U_1 V_2 V_3 \ldots V_N, \tag{82}$$

with $U_1 = \exp(i\frac{\chi}{2}\sigma_z^{(1)}\sigma_z^{(1)})$, $V_j = \exp(i\chi\sigma_z^{(1)}\sigma_z^{(j)})$ and $\chi = -2gt$. We demonstrate that in the limit of a large number of particles N (equivalent to the limit of weak or short-time interactions expressed by the relation $\chi = \frac{\theta}{N-1}$ with a certain finite constant θ), the dynamics of a single qubit resulting from pairwise interactions is equivalent to

$$\lim_{N\to\infty} V_2V_3\dots V_N = e^{-igt\langle\sigma_z\rangle\sigma_z/2}, \tag{83}$$

while

$$\lim_{N\to\infty} U_1 = \lim_{N\to\infty} e^{i\frac{\theta}{2(N-1)}\sigma_z^{(1)}\sigma_z^{(1)}} = \mathbb{1}. \tag{84}$$

Therefore, we see that each of the interacting qubits undergoes effective nonlinear dynamics governed by the evolution operator $U(\rho) = e^{-i\beta/2\langle\sigma_z\rangle\sigma_z}$, with $\langle\sigma_z\rangle = \mathrm{Tr}(\rho\sigma_z)$ and $\beta = g\tau$, where τ is the duration of the interaction.

In our model, we consider a situation where after each interaction time τ, we apply an external field to perform a rotation of each qubit around the y-axis, and then we apply the action of amplitude damping channel, given by Kraus operators $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{r} \end{pmatrix}$ and $K_2 = \begin{pmatrix} 0 & \sqrt{1-r} \\ 0 & 0 \end{pmatrix}$, which operations define the mapping in the form

$$\rho_{t+1} = K_1V(\rho)\rho_tV^\dagger(\rho)K_1^\dagger + K_2V(\rho)\rho_tV^\dagger(\rho)K_2^\dagger, \tag{85}$$

with evolution operator $V(\rho) = e^{-i\frac{\alpha}{2}\sigma_y}U(\rho)$ generating the standard dynamics of the kicked rotor. The state of the qubit after t steps of evolution is given by $\rho_t = \frac{1}{2}(\mathbb{1} + x_t\sigma_x + y_t\sigma_y + z_t\sigma_z)$, with the components of the Bloch vector $\mathbf{v}_t$ given by

$$\begin{aligned} x_{t+1} &= \sqrt{r}[(x_t\cos(\beta z_t) - y_t\sin(\beta z_t))\cos\alpha + z_t\sin\alpha], \\ y_{t+1} &= \sqrt{r}[x_t\sin(\beta z_t) + y_t\cos(\beta z_t)], \\ z_{t+1} &= 1 + r[(y_t\sin(\beta z_t) - x_t\cos(\beta z_t))\sin\alpha + z_t\cos\alpha - 1]. \end{aligned} \tag{86}$$

The search for fixed points of the model, $\mathbf{v}^*$, amounts to finding the zeros of the function

$$\begin{aligned} f(z^*, r, \alpha, \beta) &= -z^* + 1 + rz^*\cos(\alpha) - r \\ &\quad + rz^*\sin^2(\alpha)\frac{r - \sqrt{r}\cos(\beta z^*)}{1 + r\cos(\alpha) - \sqrt{r}(\cos(\alpha)+1)\cos(\beta z^*)}, \end{aligned} \tag{87}$$

and we proceed to find these fixed points using numerical methods, by arbitrarily setting the parameter values.

As known from [R64], for the parameter values $\alpha = \frac{\pi}{2}, \beta = 6$, the evolution generated by the operator V in the classical limit leads to chaotic dynamics of the kicked rotor. In such a case, the presented model will be described by a single parameter r representing the strength of the amplitude damping channel. Notice that in the extreme case of $r = 0$, the state of the system is projected onto $|0\rangle$ already in the first step of evolution, while for $r = 1$, there is no damping, and thus the evolution reduces to the dynamics of the chaotic kicked rotor. Therefore, in the range of parameters $0 < r < 1$, different regimes of the model's behavior can be expected.

From the analysis of the function (87), we find that in the range $0 \le r \le r_b \approx 0.9719$, the model exhibits one fixed point $\mathbf{v}_0$, while for $r_b \le r < 1$, there are three distinct fixed points $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$. The stability analysis of the fixed points is carried out using the Jacobian determined by linearizing the mapping (86) around the fixed points, in the form of

$$\mathcal{A}_{\mathbf{v}^*} = \begin{pmatrix} 0 & 0 & -\sqrt{r} \\ -\sqrt{r}\sin(\beta z^*) & \sqrt{r}\cos(\beta z^*) & -\beta\sqrt{r}\left(y^*\sin(\beta z^*) + x^*\cos(\beta z^*)\right) \\ r\cos(\beta z^*) & r\sin(\beta z^*) & \beta r\left(y^*\cos(\beta z^*) - x^*\sin(\beta z^*)\right) \end{pmatrix}. \tag{88}$$

For $r \le r_1 \approx 0.3138$, the fixed point $\mathbf{v}_0$ is a stable fixed point, whereas for $r > r_1$, it becomes unstable. For $r > r_b$, new fixed points appear, with $\mathbf{v}_1$ being a stable fixed point, while $\mathbf{v}_2$ is an unstable fixed point.

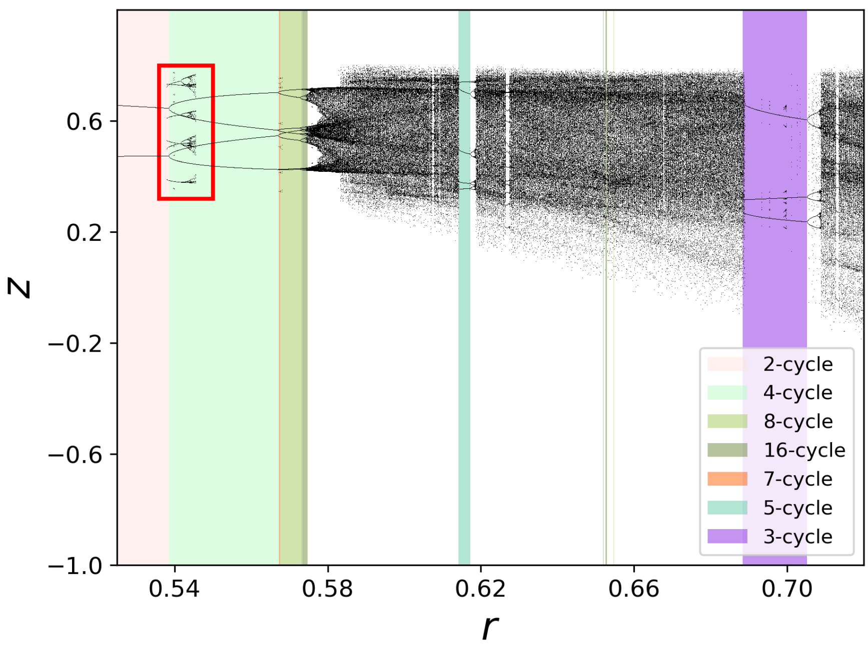

Figure 5: The bifurcation diagram for the z component of the Bloch vector with respect to the damping parameter r, with depicted regions of oscillations corresponding to the Sharkovsky ordering.

For the range $r_1 < r < r_b$, the model does not have any stable fixed points. At $r = r_1$, we observe a bifurcation of the fixed point, and oscillations of order 2 appear, leading to the qubit dynamics being restricted to jumps between two states within the Bloch sphere. As we increase the parameter r, higher-order oscillations 2^k emerge. By determining the subsequent values r_k at which further bifurcations occur, leading to the appearance of oscillations of order 2^k, we demonstrate that the ratio of consecutive differences approaches the Feigenbaum constant

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots, \tag{89}$$

being a universal behavior characterizing many nonlinear models [R71]. The bifurcation diagram for the Bloch vector component z is shown in Figure 5, where successive changes in the order of oscillations, consistent with the Sharkovsky ordering $1 \prec 2 \prec 4 \prec 8 \prec \cdots \prec 7 \prec 5 \prec 3$, are depicted. Odd-order oscillations appear in periodic windows within the chaotic behavior.

After crossing the value $r_\infty \approx 0.578$, the system's behavior becomes chaotic, while for parameter values $r_b < r < 1$, the system returns to stable behavior. For the interval $r_\infty < r < r_b$, except for the oscillation windows, the asymptotic dynamics of the Bloch vector takes place on a strange attractor, forming a specific subset of the Bloch sphere, as shown in Figure 6. By numerically determining the correlation dimension for 10,000 steps of evolution from random initial states $\mathbf{v}_0$, we obtain the fractal dimension of the strange attractor, which has a value of $d_f = 1.84$ for the specified parameter $r = 0.75$ (Figure 7). It is worth noting that the bifurcation diagram exhibits the appearance of self-similar structures within the interval $r < r_\infty$, which characterize the quantum model considered in our work and do not have an equivalent in the universal Feigenbaum bifurcation scheme present in one-dimensional classical models.

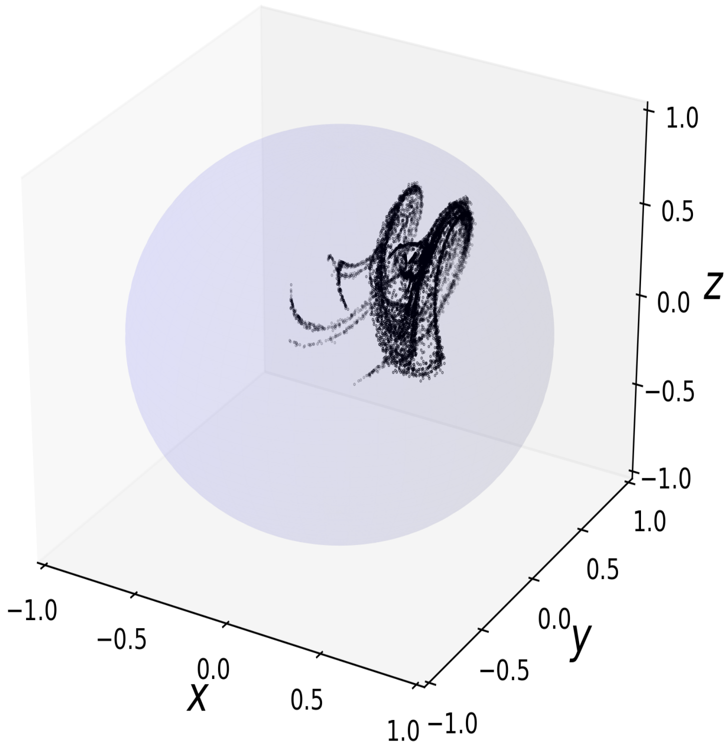

Figure 6: Visualization of a strange attractor obtained from 10,000 steps of evolution starting from a random initial state $\mathbf{v}_0$ with a damping parameter value of $r = 0.75$. The Bloch vector coordinates are marked with black dots.

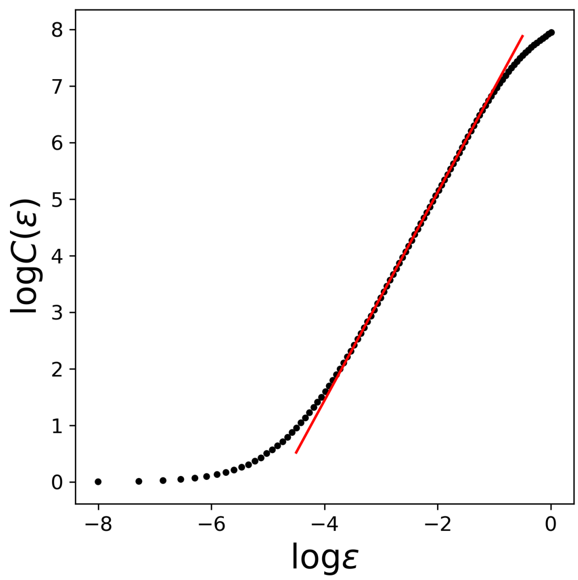

Figure 7: Estimation of the fractal dimension of the strange attractor for $r = 0.75$. For a chosen point on the attractor $\mathbf{w}$, we define a ball of radius ϵ, and then, for a given radius, we count the number of points after 10,000 steps of evolution that are contained within the ball. Averaging over different choices of $\mathbf{v}_0$ and $\mathbf{w}$, the number of points $C(\epsilon)$ scales with the ball radius as $C(\epsilon) \propto \epsilon^{d_f}$, where d_f is the fractal dimension value, which can be obtained from the slope of the best linear fit to the linear part of the dependence of $\ln C$ on $\ln \epsilon$.

5 Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

5.1 Teaching achievements

5.1.1 Doctoral dissertation supervision

Assistant Supervisor in the PhD thesis of Mahasweta Pandit, doctoral dissertation titled *'Characterization of quantum correlations with strong non-classical properties'*, defence of the doctoral dissertation: 7.07.2022, Gdańsk.

5.1.2 Academic teaching

Academic year 2022/2023:

- *Theoretical and Practical Foundations of Higher Mathematics* – lecture (25 hours) and tutorials (35 hours) for the Physics major, 1st year of undergraduate studies, University of Gdańsk
- *Quantum Physics* – tutorials (45 hours) for the Physics major, 1st year of graduate studies, University of Gdańsk
- *Vector Analysis* – lecture (45 hours) and tutorials (45 hours) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk
- *Discrete Probability Calculus* – tutorials (15 hours, 2 groups) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk
- *Linear Algebra* – tutorials (30 hours) for the Medical Physics major, 1st year of undergraduate studies, University of Gdańsk.
- *Introductory Mathematics Course* (15 hours) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk

Academic year 2021/2022:

- *Analytical Mechanics* – lecture (30 hours) and tutorials (30 hours) for the Physics major, 2nd year of graduate studies, University of Gdańsk
- *Theoretical Foundations of Higher Mathematics* – lecture (25 hours) for the Physics major, 1st year of undergraduate studies, University of Gdańsk
- *Quantum Physics* – tutorials (45 hours) for the Physics major, 1st year of graduate studies, University of Gdańsk
- *Vector Analysis* – lecture (45 hours) and tutorials (45 hours) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk
- *Discrete Probability Calculus* – tutorials (15 hours, 2 groups) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk
- *Introductory Mathematics Course* (30 hours) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk

Academic year 2020/2021:

- *Vector Analysis* – lecture (45 hours) and tutorials (45 hours) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk

- *Discrete Probability Calculus* – tutorials (15 hours, 2 groups) for the Bioinformatics major, 1st year of undergraduate studies, University of Gdańsk

Academic year 2013/2014:

- *Mechanics* – tutorials (30 hours) and computer laboratory (30 hours) for the Physics major, 1st year of master's studies, Adam Mickiewicz University
- *Mechanics* – tutorials (30 hours) for the Biophysics major, 1st year of master's studies, Adam Mickiewicz University

Academic year 2012/2013:

- *Mechanics* – tutorials (30 hours) for the Physics major, 1st year of master's studies, Adam Mickiewicz University
- *Mechanics* – tutorials (30 hours) for the Biophysics major, 1st year of master's studies, Adam Mickiewicz University

Academic year 2011/2012:

- *Information Technologies* – computer laboratory (30 hours) for the Philosophy major, 1st year of master's studies, Adam Mickiewicz University
- *Multimedia Communication* – computer laboratory (30 hours) for the Philosophy major, 1st year of master's studies, Adam Mickiewicz University

Academic year 2010/2011:

- *Information Technologies* – computer laboratory (30 hours) for the Philosophy major, 1st year of master's studies, Adam Mickiewicz University

5.2 Organizational achievements

- Chairman of the Admissions Committee at the Faculty of Mathematics, Physics and Informatics, University of Gdańsk, year 2023.
- Chairman of the Admissions Committee for the studies in the *Quantum information technology* program at the Faculty of Mathematics, Physics and Informatics, University of Gdańsk, year 2022.

5.3 Achievements in popularization of science

- Popular science lecture: *Quantum Phenomena – From Cakes Measurements to Teleportation*, delivered at the High School of Good Education Academy in Gdańsk, March 2020.
- Popular science lecture: *Quantum Connections, or What Sets Light and Atoms Apart from Balls, Coins, and Socks*, delivered at the High School of Good Education Academy in Gdańsk, March 2019.
- Participation in the preparation of the television program *Scientific Shot – Quantum Information*, September 2011.

6 Other scientific achievements

6.1 Awards

- Individual Second Degree Rector's Award of the University of Gdańsk, year 2022.
- Scholarship of the City of Poznań for young researchers from the academic community in Poznań, year 2013.
- Group Second Degree Rector's Award of Adam Mickiewicz University, year 2013.

6.2 Additional scientific research achievements after obtaining the doctoral degree

- **Detection of multipartite entanglement using random measurements**
  In the paper [O1], we present a scheme for detecting genuine multipartite entanglement in the absence of a distinguished reference frame with respect to which measurement bases are defined. The starting point of the proposed method is the measurement of the multipartite system with random local measurement settings, followed by the analysis of the full and partial rank tensor correlations using the second moments of relevant correlation element distributions. We introduce analytically derived entanglement criteria for two qubits and also present strong evidence for the existence of a genuine multipartite entanglement witness for three and four qubits.

- **Majorization uncertainty principles**
  The starting point of our work [O2] was an attempt to extend the universal uncertainty principle [R93], based on the concept of probability distribution majorization, to the case of an uncertainty principle that takes into account side information. To achieve this, we introduced an operational framework in which we formalized the notion of conditional majorization, which allowed us to define the principle of conditional uncertainty. We considered the case of classical memory and constructed conditional uncertainty principles that provided lower bounds on the minimal Bob's uncertainty for a pair of incompatible observables of Alice, conditioned on any measurement that Bob performs on his system.

- **Contextuality in the framework of resource theories**
  In the paper [O3], we introduced the concept of contextuality based on axioms known from the theory of entanglement. Specifically, we defined a class of operations that transform families of probability distributions (generally considered as contextual systems) and provided axioms that measures quantifying contextuality must satisfy. Additionally, we proved the property of asymptotic continuity of the measure of contextuality defined by the relative entropy of contextuality [O4].

- **Analysis of the effects of magnetic interactions in the Hubbard model**
  The papers [O5, O6] continue the research undertaken in earlier years on the analysis of the extended Hubbard model in the narrow-band limit [O7, O8, O9]. In the paper [O5], we analyze a one-dimensional chain model with on-site interactions and nearest-neighbor Ising-type charge and magnetic interactions. Utilizing the transfer matrix method, we find exact solutions for the model in the form of a sum of states, which allows for the analysis of the thermodynamic properties of the system. The paper [O6] is dedicated to the analysis of the influence of different types of magnetic exchange interactions on the stability of ordered phases present in the model. For this purpose, we employ various analytical and numerical methods, including mean-field variational method, Monte Carlo numerical methods, as well as exact methods in specific cases. We determine phase diagrams indicating the regions of stability for ordered phases under different parameter values of the model and different dimensionalities of the lattice. Additionally, we identify

the conditions for the stability of phase separation, i.e., the stable coexistence of two domains with a specific order parameter value.

6.3 Scientific achievements from the period preceding the conferral of the doctoral degree

6.3.1 Research included in the doctoral thesis

- **Activation of nonlocal correlations via entanglement swapping**
  One of the first problems proposed to me in the field of quantum information theory, which was significantly different from my previous research area, was the problem of activating nonlocal quantum correlations. One approach to this problem was presented in the work [O10], where we considered multiple rounds of entanglement swapping in a chain of two-qubit states. Initially, we required that each state in the chain does not allow for the violation of the CHSH inequality. We showed that after performing entanglement swapping processes on each pair of neighboring states in the chain (for a sufficiently large number of pairs of quantum states), it is possible to obtain a state of particles at the ends of the chain whose correlations allow for the violation of the CHSH inequality. In the study, we analyze the conditions (Bell measurement results, detailed properties of the quantum states of particle pairs constituting the chain) under which the activation of nonlocal correlations is possible. We also demonstrate that entanglement swapping using only one class of quantum states does not lead to the activation of nonlocal correlations.

- **Measures of contextuality**
  Unlike classical systems, quantum correlations manifest in the form of quantum contextuality, where measurement outcomes of different observables cannot be described by a single joint probability distribution. In other words, the joint probability distribution that correctly describes the statistics of measurement outcomes for a certain set of compatible observables (a context) cannot describe the statistics of outcomes obtained for a different context. Therefore, quantum contextual systems exhibit the property of perceived correlations between the choice of context and the family of probability distributions describing the statistics for all possible measurement contexts. In the paper [O4], we proposed a measure of contextuality for contextual systems called contextual mutual information, which quantitatively determines the value of these correlations. Additionally, we independently defined a measure known as the relative entropy of contextuality, which determines the statistical distance of the considered contextual system (family of probability distributions) from the nearest classical joint probability distribution. Surprisingly, we show that both measures of contextuality are equal, as demonstrated in the paper. Furthermore, we calculate the values of the contextuality measures for essential contextual systems: Peres-Mermin square, Mermin star, KCBS system, Popescu-Rohrlich system, as well as for the entire class of contextual systems represented by the chain Bell inequalities.

- **The relationship between post-quantum correlations and random access codes**
  In the paper [O11], we analyze the possibility of interchangeability of two post-quantum resources. One of them is random access code, a hypothetical system that allows delivering information about the value of either of two bits sent by the sender using just one transmitted bit of information. The second resource is the Popescu-Rohrlich system, whose correlations enable the breaking of the CHSH inequality up to its maximum algebraic value. It is known that using the Popescu-Rohrlich system along with an additional transmitted bit to the receiver, one can simulate the operation of the random access code. Thus, a natural question arises: is the reverse simulation also possible? In other words, can the random access code produce correlations that characterize the Popescu-Rohrlich system? In the paper, we introduce the concept of a random access code system, which,

together with the transmitted bit of information, acts as a random access code. We then demonstrate that if this system satisfies the no-signaling condition, it can simulate the operation of the Popescu-Rohrlich system, which proves that the two post-quantum systems are equivalent. In the subsequent part, we formulate an inequality connecting different resources, showing, among other things, that random access code along with 1 bit of shared randomness allows obtaining correlations characteristic of the Popescu-Rohrlich system and additionally achieving an erasure channel.

- **Information exclusion principles**
  In the paper [O12], we analyze the information exclusion relations that constrain the sum of two mutual informations for relevant pairs of observables measured on arbitrary two-qubit states. Initially, we focus on the weakness of Hall's information exclusion relation [R94], which directly arises from the entropic uncertainty principle of Maassen and Uffink [R95], linked to the fact that for observables with a common eigenvector, the Maassen-Uffink principle becomes vacuous. We introduce an information exclusion relation for the case where one party measures only one observable, while the other measures one out of two observables, and demonstrate that it is stronger than Hall's principle. We provide a proof of this relation for specific cases. Additionally, we introduce an information exclusion relation that limits the mutual information when both parties measure one out of two observables, and we present a proof of this relation for a certain class of quantum states.

6.3.2 Research not included in the doctoral thesis

- **Detection of quantum entanglement in electron spins using ferromagnetic contacts**
  The work [O13] (also [O14]) is devoted to the analysis of using imperfect ferromagnetic detectors for the detection of quantum entanglement and their utility for implementing basic quantum protocols. As a starting point, we present a model of measurements using imperfect detectors, which is equivalent to ideal measurements performed on a system subjected to the action of a depolarizing channel. We investigate the minimum value of spin polarization of the ferromagnet that determines the efficiency of the detectors, for which the detection of entanglement between two spins is possible based on different entanglement witnesses. We also determine the conditions under which imperfect ferromagnetic detectors can be used in the SIMCAP cryptographic key distillation protocol, as well as for violating the CHSH inequality. By adapting a phenomenological model of spin dephasing, we also analyze the possibilities of detection, taking into account the effects of spin relaxation and dephasing of electron spins in the transport channel.

- **Popular physics topics**
  The work [O15] presents an analysis of the motion of a material point moving along a curved path, taking into account friction. Specifically, we consider motion on paths in the shape of vertically oriented circles and ellipses, where we determine the minimum initial velocity required to traverse the full length of the path. For the circular path, we find an exact solution, while in general, we determine the values of velocity and friction coefficient for which the motion ceases or the material point loses contact with the path.

  The work [O16] is devoted to discussing a method for constructing networks of unit resistors whose equivalent resistance assumes a specific value. It is observed that the scheme of connecting successive segments of several resistors in an alternating parallel and series manner reproduces the equivalent resistance given by the corresponding continued fraction. This straightforwardly enables the construction of a circuit whose equivalent resistance can take any real value with a specified accuracy.

- **Information flow of quantum states interacting with closed timelike curves**
  The interaction of quantum systems with closed timelike curves, whose existence is suggested by certain solutions of the general relativity theory, is typically described in the framework of the non-linear Deutsch model [R96]. The Ralph-Myers circuit model [R97], providing an alternative description of closed timelike curves, seems to indicate their strong potential, including the ability to distinguish non-orthogonal quantum states. In the commentary [O17], we demonstrate that this conclusion is based on an unclear assumption regarding the description of the resource used in the circuit model. Specifically, we show that different forms of classically correlated states with certain properties used as resources can lead to fundamentally different results.

- **Magnetic and charge orderings in the extended Hubbard model**
  The papers [O7, O8, O9] present an analysis of the Hubbard model in the narrow band limit with effective on-site and inter-site magnetic and charge interactions. Depending on the nature and strength of these interactions, phase diagrams in the mean-field approximation are obtained. The work [O7] is devoted to the analysis of magnetic interactions. We show for which parameter values of the model the ground state is realized in a magnetically ordered or disordered phase. We also find conditions under which phase separation is possible, i.e., stable coexistence of two domains with different charge concentrations. In the work [O8], we consider charge interactions including nearest and next-nearest neighbors. Depending on the parameter values of the model, in addition to the homogeneous stable phases (charge ordered and disordered), phase separation is possible, both charge-ordered-disordered and separation of two charge-ordered phases with different charge concentrations. In the work [O9], we consider the effects of both magnetic and charge interactions in the limit of strong on-site repulsive potential. Despite its simplicity, the model exhibits complex behavior, both in terms of the possibility of multiple types of stable ordered phases (including magnetic, charge, mixed orderings, and phase separations of different kinds), as well as in terms of multi-critical behaviors characterizing the phase diagrams.

6.4 Scientometric information

- The number of peer-reviewed publications: 24 (11 before obtaining the doctoral degree)
- The number of unpublished manuscripts in the arXiv database: 3 (currently under review in scientific journals)
  1) `https://arxiv.org/abs/2208.01983`
  2) `https://arxiv.org/abs/2209.05197`
  3) `https://arxiv.org/abs/2303.09238`
- Number of citations (source: **Google Scholar**, 1.08.2023)
  - Total number of citations: 402
  - H-index: 11
  - I10-index: 13
- Number of citations (source: **Web of Science**, 1.08.2023)
  - Total number of citations: 300
  - Total number of citations (without self-citations): 285
  - H-index: 10

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[H2] Justyna Łodyga, Waldemar Kłobus, Ravishankar Ramanathan, Andrzej Grudka, Michał Horodecki, and Ryszard Horodecki. Measurement uncertainty from no-signaling and nonlocality. *Phys. Rev. A*, 96:012124, Jul 2017.

[H3] Waldemar Kłobus, Marek Miller, Mahasweta Pandit, Ray Ganardi, Lukas Knips, Jan Dziewior, Jasmin Meinecke, Harald Weinfurter, Wiesław Laskowski, and Tomasz Paterek. Cooperation and dependencies in multipartite systems. *New Journal of Physics*, 23(6):063057, jun 2021.

[H4] Waldemar Kłobus, Wiesław Laskowski, Tomasz Paterek, Marcin Wieśniak, and Harald Weinfurter. Higher dimensional entanglement without correlations. *Eur. Phys. J. D*, 73:29, 2019.

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