Summary of Professional Accomplishments

Dr Adam Kwela

Contents

1 Name

Adam Kwela

2 Diplomas, degrees conferred in specific areas of science or arts, including the name of the institution which conferred the degree, year of degree conferment, title of the PhD dissertation

PhD in Mathematics
Institution: Institute of Mathematics of the Polish Academy of Sciences
Title of PhD dissertation: Combinatorial and descriptive properties of ideals on countable sets
Supervisor: Prof. Dr hab. Piotr Zakrzewski
Auxiliary supervisor: Dr Marcin Sabok
Date: October 24, 2014

MSc in Mathematics
Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
Title of Master's Thesis: Rudin-Keisler order
Supervisor: Prof. Dr hab. Ireneusz Recław
Date: July 8, 2010

3 Information on employment in research institutes or faculties/departments or school of arts

since 01.01.2015: Adiunkt in the Institute of Mathematics, University of Gdańsk

4 Description of the achievements, set out in art. 219 para. 1 point 2 of the Act

4.1 Series of scientific articles: "Set-theoretic and topological applications of the Katětov order".

The habilitation series consists of 7 scientific articles (in the order they will be discussed below):

[H1] Adam Kwela, *Unboring ideals*, Fund. Math. **261** (2023), no. 3, 235–272.

[H2] Rafał Filipów, Krzysztof Kowitz, Adam Kwela, and Jacek Tryba, *New Hindman spaces*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 891–902. MR 4356195

[H3] Rafał Filipów, Krzysztof Kowitz, and Adam Kwela, *Characterizing existence of certain ultrafilters*, Ann. Pure Appl. Logic **173** (2022), no. 9, Paper No. 103157. MR 4448270

[H4] Adam Kwela, *On extendability to F_σ ideals*, Arch. Math. Logic **61** (2022), no. 7-8, 881–890. MR 4495906

[H5] Rafał Filipów and Adam Kwela, *Yet another ideal version of the bounding number*, J. Symb. Log. **87** (2022), no. 3, 1065–1092. MR 4472525

[H6] Adam Kwela, *On a conjecture of Debs and Saint Raymond*, Fund. Math. **260** (2023), no. 1, 67–76. MR 4516186

[H7] Adam Kwela and Paolo Leonetti, *Density-like and generalized density ideals*, J. Symb. Log. **87** (2022), no. 1, 228–251. MR 4404626

Throughout the self-presentation, bibliographic items [H1]-[H7] refer to the above series, items [O1]-[O20] refer to my other publications (the full list of which, together with their description, can be found in section 4.2), while the remaining items ([E1]-[E∞]) are articles by other authors and are listed at the end of the self-presentation.

4.1.1 Introduction

A family $\mathcal{I} \subseteq \mathcal{P}(M)$ is called an ideal on the set M if it is closed under taking subsets and finite unions of its elements. If $\mathcal{I}$ is an ideal on M, then $\mathcal{I}^\star = \{A \subseteq M : \ M \setminus A \in \mathcal{I}\}$ is its dual filter (a family closed under taking supersets and finite intersections of its elements), and $\mathcal{I}^+ = \{A \subseteq M : \ A \notin \mathcal{I}\}$ is the family of $\mathcal{I}$-positive sets. In what follows, when I use the word "ideal", I will additionally assume that it is an ideal containing $[M]^{<\omega}$ (so $\bigcup \mathcal{I} = M$) and satisfying $M \notin \mathcal{I}$. Moreover, unless specifically stated, I will only consider ideals on infinite countable sets. For greater clarity, I will formulate some definitions and results only for ideals on ω (for any infinite countable set M one can find its bijection with ω and, using it, "move" the ideal from M to ω, preserving all of its needed properties).

By identifying subsets of M with their characteristic functions, we can treat ideals on M as subsets of the topological space 2^M with the product topology (where on $2 = \{0, 1\}$ we consider the discrete topology), so we can talk, for example, about ideals of Borel class $\mathbf{\Sigma^0_2}$ or ideals with the Baire property.

For ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$. Although the relation $\leq_K$ is only a quasi-order, it is often called Katětov order. M. Katětov introduced it in the 1960s in [E64], and then used it in [E65] and [E66] to study Baire classes with respect to ideal convergence (see subsection 4.1.7). Since then, the Katětov order has been extensively studied both as a convenient tool (e.g. for classifying undefinable objects such as ultrafilters with the use of Borel ideals – see [E54] and subsection 4.1.4) and itself, as a relation on a set of ideals (e.g. H. Sakai in [E83] showed that $\leq_K$ on Borel ideals is σ-directed). In recent years, a group of mathematicians related to M. Hrušák has made a particularly significant contribution to the development of our knowledge of the Katětov order – e.g. O. Guzmán-González and D. Meza-Alcántara showed in [E50] that even among P-ideals of the Borel class $\mathbf{\Sigma^0_2}$ there are antichains of length $\mathfrak{c}$ and chains of length $\mathfrak{b}$ in the Katětov order ($\mathcal{I}$ is a P-ideal if for each sequence $(A_n)_{n\in\omega} \in \mathcal{I}^\omega$ there is a set $A \in \mathcal{I}$ such that $|A_n \setminus A| < \omega$ for all $n \in \omega$).

We say that an ideal $\mathcal{I}$ on M is tall if in every $A \in [M]^\omega$ one can find $B \in [A]^\omega \cap \mathcal{I}$. For an ideal $\mathcal{I}$ and a set $A \in \mathcal{I}^+$ we define the ideal:

$$\mathcal{I} \restriction A = \{A \cap B : \ B \in \mathcal{I}\}.$$

Examples of ideals studied in the literature that will be of particular interest to us in this self-presentation are:

- $\mathrm{Fin} = \{A \subseteq \omega : \ |A| < \omega\}$ (non-tall P-ideal of the Borel class $\mathbf{\Sigma^0_2}$),
- $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \ \sum_{n\in A} \frac{1}{n+1} < \infty\right\}$ (tall P-ideal of the Borel class $\mathbf{\Sigma^0_2}$),
- $\mathcal{I}_d = \left\{A \subseteq \omega : \ \lim_{n\to\infty} \frac{|A\cap n|}{n+1} = 0\right\}$ (tall P-ideal of the Borel class $\mathbf{\Pi^0_3}$; here I use the set-theoretic convention: $n = \{0, 1, \ldots, n-1\}$),
- $\mathrm{Fin}^2 = \left\{A \subseteq \omega^2 : \ \{n \in \omega : \ A_{(n)} \notin \mathrm{Fin}\} \in \mathrm{Fin}\right\}$, where $A_{(n)} = \{k \in \omega : \ (n, k) \in A\}$ (tall non-P-ideal of the Borel class $\mathbf{\Sigma^0_4}$),
- conv consisting of those $A \subseteq [0, 1] \cap \mathbb{Q}$ that have only finitely many cluster points (tall non-P-ideal of the Borel class $\mathbf{\Sigma^0_4}$).

It is easy to see that an ideal $\mathcal{I}$ is tall if and only if $\mathcal{I} \not\leq_K \mathrm{Fin}$. In subsections 4.1.2, 4.1.3 and 4.1.4, we will look for similar characterizations (also using the Katětov order) of the existence of certain sequentially compact spaces or the existence of certain classes of ultrafilters. In subsections 4.1.5 and 4.1.7, we will show the limitations of this approach, answering questions of M. Hrušák and of G. Debs and J. Saint Raymond. In turn, in subsections 4.1.6 and 4.1.8, we will deal with issues not directly related to the Katětov order (e.g. we will solve a problem posed by P. Borodulin-Nadzieja, B. Farkas and G. Plebanek, concerning the existence of certain P-ideals), but in the proofs of some theorems we will use the results concerning this order.

4.1.2 Sequentially compact spaces with respect to an ideal

In the article [H1] I dealt with some natural modification of sequential compactness that gives control over the size of the convergent subsequence: if $\mathcal{I}$ is an ideal on ω, then we say that a topological space X is $\mathrm{FinBW}(\mathcal{I})$ if each sequence $(x_n)_{n\in\omega}$ in X has a convergent subsequence $(x_n)_{n\in A}$ indexed by some $A \in \mathcal{I}^+$. Of course, $\mathrm{FinBW}(\mathrm{Fin})$ is the class of sequentially compact spaces, and for each ideal $\mathcal{I}$ and each topological space X we have:

$$X \text{ is finite } \implies X \text{ is } \mathrm{FinBW}(\mathcal{I}) \implies X \text{ is } \mathrm{FinBW}(\mathrm{Fin}).$$

Spaces of this type have been studied for over twenty years in the context of several specific definable ideals, mainly related to infinite combinatorics and Ramsey theory, e.g. by M. Kojman ([E68]) and J. Flašková ([E46]) – this is described in detail in subsection 4.1.3. On the other hand, in the papers [E41] and [E43], ideals $\mathcal{I}$, for which the interval $[0,1]$ is $\mathrm{FinBW}(\mathcal{I})$, were thoroughly investigated. As observed by D. Meza-Alcántara in [E75, Section 2.7], such ideals have a nice characterization using Katětov order: $[0,1]$ is $\mathrm{FinBW}(\mathcal{I})$ if and only if $\mathrm{conv} \not\leq_K \mathcal{I}$.

In [H1], I defined a new ideal $\mathcal{BI}$ of class $\mathbf{\Sigma^0_4}$, such that for each ideal $\mathcal{I}$ the following implications (marked with arrows) hold:

$$\begin{tikzcd}
\mathcal{I} \text{ is extendable to a } \mathbf{\Sigma^0_2} \text{ ideal} \arrow[d, "\text{obvious}"] \arrow[dr, "\text{[E41, 3.4]+[E75, Sec. 2.7]}"] & \\
\mathcal{I} \text{ is extendable to a } \mathbf{\Pi^0_4} \text{ ideal} \arrow[d, "\text{[E11]+[E33]}^{1}"] & \mathrm{conv} \not\leq_K \mathcal{I} \arrow[d, "\text{[H1, 4.6]}"] \\
\forall_{A\in\mathcal{I}^+}\ \mathrm{Fin}^2 \not\leq_K \mathcal{I}\restriction A \arrow[r, "\text{[H1, 4.10]}"] & \mathcal{BI} \not\leq_K \mathcal{I} \arrow[d, "\text{[H1, 4.6]}"] \\
& \mathrm{Fin}^2 \not\leq_K \mathcal{I}
\end{tikzcd}$$

Moreover, for each ideal $\mathcal{I}$ we have:

(a) [H1, Theorem 6.5]: $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$ if and only if there exists an infinite Hausdorff $\mathrm{FinBW}(\mathcal{I})$ space.

(b) [H1, Proposition 6.2 and 6.3]: $\mathcal{BI} \not\leq_K \mathcal{I}$ if and only if there is a Hausdorff $\mathrm{FinBW}(\mathcal{I})$ space that is not a disjoint union of finitely many one-point compactifications of discrete spaces.

(c) [H1, Theorem 6.6]: Assuming the continuum hypothesis (CH), $\mathcal{BI} \not\leq_K \mathcal{I}$ if and only if there exists an uncountable separable Hausdorff space which is $\mathrm{FinBW}(\mathcal{I})$ (the implication "$\Leftarrow$" is true in ZFC).

(d) [H1, Proposition 6.1]: If $\mathrm{Fin}^2 \not\leq_K \mathcal{I}\restriction A$ for all $A \in \mathcal{I}^+$, then ω_1 with the order topology is $\mathrm{FinBW}(\mathcal{I})$.

The ideal $\mathcal{BI}$ also allows to answer the question posed by M. Hrušák in [E54, Question 5.18], and then repeated in [E55, Question 5.10]: Is it true that for every Borel ideal $\mathcal{I}$ we have $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$ if and only if $\mathcal{I}$ is extendable to an ideal of Borel class $\mathbf{\Pi^0_3}$? It turns out that $\mathrm{Fin}^2 \not\leq_K \mathcal{BI}$, but $\mathcal{BI}$ cannot be extended to a $\mathbf{\Pi^0_3}$ ideal ([H1, Example 7.1]).

I have also shown, assuming Martin's axiom for σ-centered partial orders (MA(σ-centered)), that if $\mathcal{I}$ is a coanalytic ideal satisfying the condition $\mathrm{Fin}^2 \not\leq_K \mathcal{I}\restriction A$ for all $A \in \mathcal{I}^+$ (in particular, if $\mathcal{I}$ is of class $\mathbf{\Pi^0_4}$), then for every ideal $\mathcal{J}$ we have: $\mathcal{J} \not\leq_K \mathcal{I}$ if and only if there is a Hausdorff

[1] In [H1, Proposition 4.9] it is explained in detail how this implication follows from [E11] and [E33].

FinBW($\mathcal{I}$) space that is not FinBW($\mathcal{J}$) (the implication "$\Leftarrow$" is true in ZFC, and the proof of "$\Rightarrow$" uses a certain ideal game described in subsection 4.1.6). On the other hand, the above result cannot be extended to all ideals $\mathcal{I}$, because there is an ideal $\mathcal{I}$ of class $\mathbf{\Sigma^0_4}$, such that $\mathcal{I}_d \not\leq_K \mathcal{I}$, but any FinBW($\mathcal{I}$) space is also FinBW($\mathcal{I}_d$) ([H1, Example 10.6]).

4.1.3 Hindman spaces

For $D \in [\omega]^\omega$ we will use the notation:

$$FS(D) = \left\{ \sum_{n \in F} n : \ F \in [D]^{<\omega} \setminus \{\emptyset\} \right\}.$$

We say that a sequence $(x_n)_{n \in FS(D)}$ in a topological space X is IP-convergent to $x \in X$ if for every open neighborhood U of the point x there exists such $k \in \omega$ that $x_n \in U$ for all $n \in FS(D \setminus k)$ (this kind of convergence was introduced by H. Furstenberg and B. Weiss in [E48] in the context of dynamical systems). In the paper [E67], M. Kojman defined Hindman spaces as those in which for each sequence $(x_n)_{n\in\omega}$ one can find such a set $D \in [\omega]^\omega$ that $(x_n)_{n \in FS(D)}$ is IP-convergent. Later, together with S. Shelah in [E69], he showed that, assuming CH, there is a non-Hindman Hausdorff FinBW($\mathcal{W}$) space, where $\mathcal{W}$ is the ideal consisting of those $A \subseteq \omega$ that do not contain arbitrarily long finite arithmetic progressions. As far as I know, the question of the existence of such a space in ZFC remains open.

In [E60] A. Jones strengthened the above result by replacing CH with MA(σ-centered), and asked if it is consistent with ZFC that there is a Hindman space which is not FinBW($\mathcal{W}$). The above question was our main motivation in the paper [H2]. Before formulating its results, we need to define the Hindman ideal:

$$\mathcal{H} = \left\{ A \subseteq \omega : \ \forall_{D \in [\omega]^\omega} \ FS(D) \not\subseteq A \right\}$$

(this family, of class $\mathbf{\Pi^1_1}$, is an ideal by Hindman's theorem).

I have significantly contributed to the following two results:

(a) [H2, Theorem 2.5]: (CH) If $\mathcal{I} \not\leq_K \mathcal{H}$, then there is a separable Hindman space that is not FinBW($\mathcal{I}$).

(b) [H2, Theorem 3.2]: $\mathcal{I}_{1/n} \not\leq_K \mathcal{H}$, so, assuming CH, there is a Hindman space that is not FinBW($\mathcal{I}_{1/n}$) – this answers the question posed by J. Flašková at the 22nd Summer Conference on Topology and its Applications (2007).

The construction of the space in (a) differs from that proposed by M. Kojman and S. Shelah in [E69] and then modified by me in [H1] (where the main part of the proof is to find an appropriate maximal almost disjoint family, and then use the one-point compactification of the Mrówka space given by this family) – we failed in our attempt to show that the resulting space is Hausdorff, but we proved that limits (as well as IP-limits) of sequences in this space are unique.

Using [H2, Proposition 1.1] and [H1, Theorem 6.6] described in the previous subsection, we get the following characterization: (CH) $\mathcal{BI} \not\leq_K \mathcal{I}$ if and only if there is a FinBW($\mathcal{I}$) space that is not Hindman ([H1, Corollary 11.5]). Moreover, if $\mathcal{I}$ is extendable to an ideal of the class $\mathbf{\Pi^0_4}$, then CH can be relaxed to MA(σ-centered) (by [H2, Proposition 1.1] and [H1, Corollary 6.8]). This extends to a larger class of ideals the previously mentioned theorem by M. Kojman and S. Shelah from [E69], as well as J. Flašková's result from [E46], where this is shown for all $\mathbf{\Sigma^0_2}$ ideals.

Returning to A. Jones' question, $\mathcal{W}$ is an ideal of the class $\mathbf{\Sigma^0_2}$, so, by [H2, Proposition 2.6], if $\mathcal{W} \leq_K \mathcal{H}$, then every Hindman space is FinBW($\mathcal{W}$). Using item (a) and assuming that the sentence "$\mathcal{W} \leq_K \mathcal{H}$" is decidable in ZFC,[2] we have reduced this question to a purely combinatorial problem (which, however, we were unable to solve): is $\mathcal{W} \leq_K \mathcal{H}$? The above reasoning is true for any definable ideal $\mathcal{I}$ of class $\mathbf{\Sigma^0_2}$ for which "$\mathcal{I} \leq_K \mathcal{H}$" is not independent of ZFC (see [H2, Corollary 2.8], which lacks the assumption that the sentence "$\mathcal{I} \leq_K \mathcal{H}$" is decidable in ZFC; see also [H1, Corollary 11.5(b)]).

[2] As far as I know, there are no known definable ideals that do not satisfy this assumption, i.e. such definable ideals $\mathcal{I}$ and $\mathcal{J}$ for which "$\mathcal{I} \leq_K \mathcal{J}$" would be independent of ZFC. For Borel ideals, "$\mathcal{I} \leq_K \mathcal{J}$" is absolute by Shoenfield's absoluteness theorem (see [E18] or [E54, Proposition 1.3]).

4.1.4 $\mathcal{I}$-ultrafilters

If $\mathcal{I}$ is an ideal, then an ultrafilter (filter maximal with respect to inclusion) $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for each function $f : \omega \to \bigcup \mathcal{I}$ there is a set $A \in \mathcal{U}$ such that $f[A] \in \mathcal{I}$ (or, equivalently, using $\leq_K$: if $\mathcal{I} \not\leq_K \mathcal{U}^\star$, where $\mathcal{U}^\star$ is the ideal dual to $\mathcal{U}$). This concept was introduced by J. Baumgartner in [E13] and since then it has been used many times in the literature, e.g. by J. Brendle and J. Flašková in [E18] or M. Hrušák in [E54].

$\mathcal{I}$-ultrafilters allow to characterize well-known families of ultrafilters with definable (usually Borel) ideals – therefore, such ideals are particularly important in the context of ultrafilters.

An example of the approach described in the previous paragraph are P-points, i.e. ultrafilters on ω dual to P-ideals. P-points were used by W. Rudin in [E81] to solve a topological problem concerning the Stone-Čech compactification of ω. As shown by S. Shelah, the existence of P-points is independent of ZFC ([E91]). M. Hrušák noticed that for any ultrafilter $\mathcal{U}$ on ω, the following conditions are equivalent (see [E18, Observation 2.1] or [E75, Theorem 2.8.7]):

- $\mathcal{U}$ is a P-point,
- $\mathcal{U}$ is a Fin^2-ultrafilter,
- $\mathcal{U}$ is a conv-ultrafilter.

It is known that if there is an $\mathcal{I}$-ultrafilter that is not a $\mathcal{J}$-ultrafilter, then $\mathcal{I} \not\leq_K \mathcal{J}$. In [H3] we explored the possibility of reversing this implication, obtaining some general results (see [H3, Theorem 4.3]; point (5) of this theorem is imprecise and should be reformulated in the spirit of [H1, Corollary 10.5(a)]). We then used our methods in the context of three well-known classes of ultrafilters: P-points, Q-points and selective ultrafilters. Below I will focus on the first one: P-points.

The first two of the following results, to which I have contributed significantly, show which features of Fin^2 and conv are crucial to the above characterization of P-points:

(a) [H3, Theorem 7.3(1)]: (CH) $\mathcal{I} \not\leq_K \mathrm{Fin}^2$ if and only if there is an $\mathcal{I}$-ultrafilter that is not a P-point (the implication "$\Leftarrow$" is true in ZFC).

(b) [H3, Theorem 6.3]: (MA(σ-centered)) A Borel ideal $\mathcal{I}$ is extendable to a $\boldsymbol{\Sigma}^0_2$ ideal if and only if there is a P-point that is not an $\mathcal{I}$-ultrafilter (the implication "$\Leftarrow$" is true in ZFC; see subsection 4.1.5 where the possibility of using $\leq_K$ in this result is investigated).

(c) [H3, Corollary 7.2(2)]: Assuming Martin's axiom for countable partial orders (MA(ctbl)), if the ideal $\mathcal{I}$ contains a tall summable ideal, then there exists an $\mathcal{I}$-ultrafilter that is not a P-point (an ideal is summable if it is of the form $\mathcal{I}_g = \left\{A \subseteq \omega : \sum_{n\in A} g(n) < \infty\right\}$ for some function $g : \omega \to [0, \infty)$ such that $\sum_{n\in\omega} g(n) = \infty$).

In order to prove item (c), we have introduced a game $G(\mathcal{I})$, associated to an ideal $\mathcal{I}$ on ω, in which Player I in his first move plays $k_0 \in \omega$, and then in the nth move Player II chooses $G_n \in \mathrm{Fin}$, while Player I in the $(n+1)$th move answers with a pair $(F_n, k_{n+1}) \in \mathrm{Fin} \times \omega$ such that $F_n \cap G_n = \emptyset$ and $|F_n| \leq k_n$. At the end, Player I is declared the winner if $\bigcup_{n\in\omega} F_n \in \mathcal{I}^+$. Otherwise, Player II is declared the winner. By Martin's theorem on Borel determinacy, this game is determined (i.e. one of the players has a winning strategy) in the case of Borel ideals.

The above game is a modification of the game $G'(\mathcal{I})$ studied by C. Laflamme in [E71] and later used by M. Laczkovich and I. Recław in [E70] (see subsections 4.1.6 and 4.1.7) – the change is in forcing Player I to declare the size (k_n) of his move (F_n) before the move of Player II (G_n). Interestingly, the existence of a winning strategy for Player II in $G(\mathcal{I})$ can be expressed in known terms: Player II has a winning strategy in $G(\mathcal{I})$ if and only if $\mathcal{I}$ contains a tall summable ideal ([H3, Lemma 5.2(2)]). On the other hand, the existence of a winning strategy for Player I can be characterized combinatorically ([H3, Lemma 5.2(1)]) in such a way that allows to prove item (c).

4.1.5 Question concerning extendability to $\boldsymbol{\Sigma}^0_2$ ideals

Comparing items (a) and (b) from subsection 4.1.4, one may want to find a characterization of extendability to a $\boldsymbol{\Sigma}^0_2$ ideal, which uses the Katětov order. In the diagram from subsection 4.1.2 there is an implication proved in [E41, Theorem 3.4]: if an ideal $\mathcal{I}$ is extendable to an ideal of class $\mathbf{\Sigma^0_2}$, then conv $\not\leq_K \mathcal{I}$. Moreover, by [E41, Theorem 4.2], in the case of analytic P-ideals it is possible to reverse the above implication. The question arises whether this is also the case for all Borel ideals. This question was asked by M. Hrušák in [E54, Question 5.16] and repeated six years later in [E55, Question 5.8].

The main result of [H4] is the negative answer to the above question – I defined the ideal $\mathrm{conv}(\mathcal{I}_d, (1/2^{n+1})_{n\in\omega})$ of Borel class $\mathbf{\Sigma^0_6}$, which cannot be extended to a $\mathbf{\Sigma^0_2}$ ideal, but conv $\not\leq_K$ $\mathrm{conv}(\mathcal{I}_d, (1/2^{n+1})_{n\in\omega})$.

The ideal conv is generated by convergent sequences – each set in conv can be covered by finitely many such sequences. The ideal $\mathrm{conv}(\mathcal{I}_d, (1/2^{n+1})_{n\in\omega})$ has a similar definition – it is generated by convergent sequences, but only those converging fast enough.[3]

The method used in [H4] does not yield a similar ideal of some lower Borel class. Thus, it remains an open problem, for example, whether in the case of $\mathbf{\Pi^0_3}$ ideals extendability to an ideal of the class $\mathbf{\Sigma^0_2}$ is equivalent to conv $\not\leq_K \mathcal{I}$ (interestingly, in [H3, Theorem 10.1] we showed that every ideal of the class $\mathbf{\Sigma^0_3}$ can be extended to a $\mathbf{\Sigma^0_2}$ ideal). Moreover, D. Meza-Alcántara asked whether for each Borel ideal $\mathcal{I}$, either conv $\leq_K \mathcal{I} \restriction A$ for some $A \in \mathcal{I}^+$, or $\mathcal{I}$ is extendable to a $\mathbf{\Sigma^0_2}$ ideal ([E75, Question 4.4.6]). This problem remains open.

4.1.6 Ideal bounding numbers

For an ideal $\mathcal{I}$ on ω and $g, h \in \omega^\omega$, we will write:

$$\mathcal{D}_\mathcal{I} = \left\{f \in \omega^\omega :\ \forall_{n\in\omega}\ f^{-1}[\{n\}] \in \mathcal{I}\right\},$$

$$g \leq_\mathcal{I} h \iff \{n \in \omega :\ g(n) > h(n)\} \in \mathcal{I}.$$

The classic bounding number $\mathfrak{b}$ is equal to the minimal cardinality of an unbounded set with respect to the relation $\leq_{\mathrm{Fin}}$ on ω^ω. It is also known that it is equal to the minimal cardinality of an unbounded set for the relation $\geq_{\mathrm{Fin}}$ on $\mathcal{D}_{\mathrm{Fin}}$.

Similar numbers for $\leq_\mathcal{I}$ on ω^ω in the context of maximal ideals (sometimes called the cofinalities of the ultrapowers $\omega^\omega/\mathcal{I}$) have been extensively studied by, for example, A. Blass and H. Mildenberger in [E14] and [E76], while in the case of Borel ideals they were investigated by B. Farkas and L. Soukup in [E39]. In the paper [H5] we examined such numbers with respect to the relationship $\geq_\mathcal{I}$ on $\mathcal{D}_\mathcal{I}$, i.e. numbers of the form:

$$\mathfrak{b}(\mathcal{I}) = \min\{|\mathcal{F}| :\ \mathcal{F} \subseteq \mathcal{D}_\mathcal{I}\ \wedge\ \forall_{g\in\mathcal{D}_\mathcal{I}}\ \exists_{f\in\mathcal{F}}\ f \not\geq_\mathcal{I} g\}$$

(in [H5] we denote them by $\mathfrak{b}(\mathcal{I},\mathcal{I},\mathcal{I})$). For maximal ideals, $\mathfrak{b}(\mathcal{I})$ was studied in the context of non-standard models of arithmetic by M. Canjar, who showed, e.g., that, assuming $\mathfrak{d} = \mathfrak{c}$, there exists a P-ideal $\mathcal{I}$ such that $\mathfrak{b}(\mathcal{I})$ is equal to the cofinality of $\mathfrak{d}$ ([E23], see also [E22]).

In this context, we were mainly interested in Borel ideals. Our motivation was the relationship (described below) with ideal QN-spaces studied, e.g., in Košice by L. Bukovský and his group. For an ideal $\mathcal{I}$ on ω we say that a sequence (x_n) of points in a topological space X is $\mathcal{I}$-convergent to $x \in X$ (we write: $x_n \xrightarrow{\mathcal{I}} x$) if for each open neighborhood U of x we have $\{n \in \omega :\ x_n \notin U\} \in \mathcal{I}$. The above convergence in the case of the ideal $\mathcal{I}_d$ is called statistical convergence and was studied among others by H. Steinhaus in [E88], and in the general case was introduced by H. Cartan in [E24] in the 1930s.

A sequence of real functions $(f_n)_{n\in\omega}$ defined on a set X:

- converges to some $f \in \mathbb{R}^X$ pointwise with respect to $\mathcal{I}$ if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for all $x \in X$,
- converges to some $f \in \mathbb{R}^X$ quasi-normally with respect to $\mathcal{I}$ if there is a sequence of positive real numbers $(\varepsilon_n)_{n\in\omega}$ such that $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ and for all $x \in X$ we have:

$$\{n \in \omega :\ |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}.$$

[3] In [H4] I call such sequences convergent $\mathcal{I}_d$-*quickly with respect to* $(1/2^{n+1})_{n\in\omega}$.

For any ideal $\mathcal{I}$ on ω, quasi-normal convergence with respect to $\mathcal{I}$ implies pointwise convergence with respect to $\mathcal{I}$. Moreover, in [E19] it was shown that quasi-normal convergence with respect to Fin is equivalent to σ-uniform convergence. In [E21], L. Bukovský, I. Recław and M. Repický studied FinQN-spaces, i.e. topological spaces that do not distinguish pointwise convergence from quasi-normal convergence with respect to Fin of sequences of continuous function.

If $\mathcal{I}$ is an ideal, then a topological space X is called an $\mathcal{I}$QN-space if every sequence of continuous functions $(f_n)_{n\in\omega} \in \left(\mathbb{R}^X\right)^\omega$ convergent to 0 pointwise with respect to $\mathcal{I}$ is also convergent to 0 quasi-normally with respect to $\mathcal{I}$. This concept was introduced in [E32]. Currently, these spaces are being thoroughly studied in Košice, e.g. by M. Repický and J. Šupina (see [E20], [E80], [E86] and [E89]). It turns out that $\mathfrak{b}(\mathcal{I})$ is the minimal cardinality of a normal space that is not an $\mathcal{I}$QN-space (it has been independently proved in [E80, Theorem 3.4] and in [H5, Section 3.4]).

In [E44, Proposition 4.1] it was shown that $\omega_1 \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{c}$ for every ideal $\mathcal{I}$. I have significantly contributed to the following results from the article [H5]:

(a) [H5, Corollary 6.9]: If $\mathcal{I}$ is an ideal of class $\mathbf{\Pi^0_4}$, then $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}$.

(b) [H5, Theorem 5.13]: Calculation of $\mathfrak{b}(\mathcal{I})$ in the case when $\mathcal{I}$ is a Fubini product of two ideals.[4]

(c) [H5, Theorem 7.4]: There is a tall ideal $\mathcal{S}$ (the so-called Solecki's ideal) of class $\mathbf{\Sigma^0_2}$, such that $\mathfrak{b}(\mathcal{S}) = \omega_1$ (a consequence of this result is that consistently there is a FinQN-space which is not an $\mathcal{S}$QN-space – this is the first known ideal with such property).

Katětov order in [H5] is not as crucial as in the previously discussed papers. However, it is implicitly used in the proof of item (a), which consists of two parts. The first is the theorem by G. Debs and J. Saint Raymond, saying that if $\mathcal{I}$ is a $\mathbf{\Pi^0_4}$ ideal, then $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$ ([E33, Theorems 7.5 and 9.1][5]). The second part is related to the ideal game $G'(\mathcal{I})$ considered by C. Laflamme in [E71] and described in subsection 4.1.4. By Martin's theorem on Borel determinacy, this game is determined in the case of Borel ideals $\mathcal{I}$. M. Laczkovich and I. Recław in [E70], using combinatorial characterizations of winnning strategies proved by C. Laflamme, noticed that Player I has a winning strategy in $G'(\mathcal{I})$ if and only if $\mathrm{Fin}^2 \leq_K \mathcal{I}$,[6] while if Player II has a winning strategy in $G'(\mathcal{I})$, then there is a set S of class $\mathbf{\Sigma^0_2}$ closed under taking subsets of its elements that separates the ideal $\mathcal{I}$ from its dual filter (i.e. $\mathcal{I} \subseteq S$ and $S \cap \mathcal{I}^\star = \emptyset$). From the existence of a winning strategy for Player II it can also be derived that $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}$. I used the same technique in [H1], in the proofs of results assuming MA(σ-centered).

4.1.7 Conjecture on separating an ideal from its dual filter by a Borel set

It is easy to show that an ideal $\mathcal{I}$ and its dual filter $\mathcal{I}^\star$ have the same descriptive complexity. Thus, if $\mathcal{I}$ is an analytic ideal, then by Suslin's separation theorem there exists a Borel set S such that $\mathcal{I} \subseteq S$ and $S \cap \mathcal{I}^\star = \emptyset$. Hence, we can define for each analytic ideal $\mathcal{I}$ its rank as:

$$\mathrm{rk}(\mathcal{I}) = \min\left\{\alpha < \omega_1 :\ \exists_{S\in\mathbf{\Sigma^0_{1+\alpha}}}\ \mathcal{I} \subseteq S \ \wedge\ S \cap \mathcal{I}^\star = \emptyset\right\}.$$

The ideal game $G'(\mathcal{I})$ described in subsections 4.1.4 and 4.1.6 allowed M. Laczkovich and I. Recław in [E70, Theorem 5] to prove that for each Borel ideal $\mathcal{I}$, the following conditions are equivalent:

- $\mathrm{rk}(\mathcal{I}) = 1$,
- for each Polish space X, we have $\mathcal{B}_1^{\mathcal{I}}(X) = \mathcal{B}_1^{\mathrm{Fin}}(X)$, where $\mathcal{B}_1^{\mathcal{I}}(X)$ is the family of all real-valued functions defined on X being a pointwise limit with respect to $\mathcal{I}$ of some sequence of continuous functions,
- $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$.

[4] For the definition of Fubini product of ideals see subsection 4.1.7.

[5] G. Debs and J. Saint Raymond used a slightly different order on ideals – the equivalence of $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$ to the original condition from [E33] is shown in [E11].

[6] Like G. Debs and J. Saint Raymond in [E33], M. Laczkovich and I. Recław used a different, but equivalent in the considered cases (by [E11]), order on ideals.

G. Debs and J. Saint Raymond independently (and using completely different methods) obtained the same result for all analytic ideals ([E33, Theorem 7.5, Corollary 7.6]) and partially generalized it to higher ranks ([E33, Theorem 3.2]): for each analytic ideal $\mathcal{I}$ and $\alpha < \omega_1$, the following conditions are equivalent:

- $\mathrm{rk}(\mathcal{I}) = \alpha$,
- for each zero-dimensional Polish space X, we have $\mathcal{B}_1^{\mathcal{I}}(X) = \mathrm{Bor}_\alpha(X)$, where $\mathrm{Bor}_\alpha(X)$ is the family of all real-valued functions on X of Borel class α.

This series of papers was inspired by M. Katětov's results from [E65] and [E66], dating back to the 1970s. Later, Baire classes with respect to ideal convergence were studied, among others, by A. Bouziad in [E17] or by T. Natkaniec and P. Szuca in [E77]. R. Filipów and P. Szuca showed in [E45] that the above characterization by M. Laczkovich and I. Recław from [E70] is also true for other types of ideal convergence.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals, then

$$\mathcal{I} \otimes \mathcal{J} = \left\{A \subseteq \bigcup \mathcal{I} \times \bigcup \mathcal{J} : \left\{n \in \bigcup \mathcal{I} : A_{(n)} \notin J\right\} \in \mathcal{I}\right\},$$

where $A_{(n)} = \{k \in \bigcup \mathcal{J} : (n,k) \in A\}$, is their Fubini product (the formula above defines an ideal also when $\mathcal{I}$ or $\mathcal{J}$, but not both at the same time, is replaced by $\{\emptyset\}$). In particular, $\mathrm{Fin}^2 = \mathrm{Fin} \otimes \mathrm{Fin}$, and going further we can recursively define $\mathrm{Fin}^{n+1} = \mathrm{Fin} \otimes \mathrm{Fin}^n$ for all $n \in \omega$.

G. Debs and J. Saint Raymond also introduced ideals Fin^α for $\omega \leq \alpha < \omega_1$, showed that $\mathrm{rk}(\mathrm{Fin}^\alpha) = \alpha$ for each $0 < \alpha < \omega_1$, and formulated the following conjecture ([E33, Conjecture 7.8]): for each analytic ideal $\mathcal{I}$ and ordinal $0 < \alpha \leq \omega$, the inequality $\mathrm{rk}(\mathcal{I}) < \alpha$ holds if and only if $\mathrm{Fin}^\alpha \not\leq_K \mathcal{I}$.[7] By the above considerations, this is true for $\alpha = 2$.[8] The proof of item (a) from subsection 4.1.6 suggests that proving this conjecture could enable us to generalize it to the class of all Borel ideals.

The main result of [H6] is that the above conjecture is no longer true for $\alpha = 3$:

$$\mathcal{CEI} = (\{\emptyset\} \otimes \mathrm{Fin}^3) \cap (\mathrm{Fin}^3 \otimes \{\emptyset\})$$

is an ideal of class $\mathbf{\Sigma}^0_6$ such that $\mathrm{Fin}^3 \not\leq_K \mathcal{CEI}$, but $\mathrm{rk}(\mathcal{CEI}) \geq 3$. Earlier, in [O11], I also proved that this conjecture is false for $\alpha = \omega$ – see subsection 4.2.3.

4.1.8 Density ideals

A function $\phi : \mathcal{P}(\omega) \to [0,\infty]$ is called a submeasure if $\phi(\emptyset) = 0$, $\phi(\{n\}) < \infty$ for each $n \in \omega$ and:

$$\forall_{A,B \subseteq \omega}\ \phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B).$$

A submeasure ϕ is lower semicontinuous if:

$$\forall_{A \subseteq \omega}\ \phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

S. Solecki in [E84, Theorem 3.1] showed that an ideal on ω is an analytic P-ideal if and only if it is of the form:

$$\mathrm{Exh}(\phi) = \left\{A \subseteq \omega : \lim_{n\to\infty} \phi(A \setminus n) = 0\right\}$$

for some lower semicontinuous submeasure ϕ such that $\omega \notin \mathrm{Exh}(\phi)$ (see also [E37, Theorem 1.2.5]). The consequence of this result is that every analytic P-ideal is of the class $\mathbf{\Pi}^0_3$.

S. Solecki and S. Todorčević in [E85] solved a problem, posed independently by J. Isbell in [E59] and by D. Fremlin in [E47], concerning a certain reduction (the so-called Tukey reduction) of the ideal of all closed nowhere dense subsets of 2^ω to the ideal $\mathcal{I}_d$. They also isolated the key property of the ideal $\mathcal{I}_d$ for this proof, and with its help they defined a new class of analytic P-ideals (for which their theorem holds): an ideal is density-like if it is of the form $\mathrm{Exh}(\phi)$ for some

[7] The original hypothesis of G. Debs and J. Saint Raymond is formulated for all ordinals $0 < \alpha < \omega_1$, but uses a different order on ideals. The equivalence of [E33, Conjecture 7.8] with the one above in the case of $0 < \alpha \leq \omega$ is a consequence of [E11].

[8] The fact that this conjecture is true for $\alpha = 1$ is proved in [E33, Proposition 7.3].

lower semicontinuous submeasure ϕ with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $(F_i)_{i\in\omega}$ is a sequence of pairwise disjoint finite subsets of ω satisfying $\phi(F_i) < \delta$, then $\phi(\bigcup_{i\in A} F_i) < \varepsilon$ for some $A \in [\omega]^\omega$.

All generalized density ideals introduced by I. Farah in [E38], i.e. ideals of the form $\mathrm{Exh}(\sup_n \phi_n)$ for some sequence $(\phi_n)_{n\in\omega}$ of submeasures with finite pairwise disjoint supports (by support of ϕ we mean $\{n \in \omega : \phi(\{n\}) > 0\}$), are density-like.

In [E16, Question 5.11], P. Borodulin-Nadzieja, B. Farkas and G. Plebanek asked whether there is a density-like ideal that is not a generalized density ideal. They were motivated by studies of ideals represented in Polish abelian groups: if X is such a group and $\mathcal{I}$ is an ideal on ω, then it is represented in X if there is a function $f : \omega \to X$ satisfying:

$$A \in \mathcal{I} \iff \sum_{n\in A} f(n) \text{ is unconditionally convergent in } X.$$

In [H7] we answered this question by giving examples of density-like ideals that are not generalized density ideals. Moreover, we have shown that there are a lot of them:

(a) [H7, Theorem 3.7]: Among ideals that are not tall, there are $\mathfrak{c}$ many pairwise non-isomorphic density-like ideals that are not generalized density ideals (ideals $\mathcal{I}$ and $\mathcal{J}$ are isomorphic – we write: $\mathcal{I} \cong \mathcal{J}$ – if there is a bijection $f : \bigcup\mathcal{J} \to \bigcup\mathcal{I}$ such that $A \in \mathcal{I} \iff f^{-1}[A] \in \mathcal{J}$ for all $A \subseteq \bigcup\mathcal{I}$).

(b) [H7, Theorem 4.24]: Among tall ideals, there are $\mathfrak{c}$ many pairwise non-isomorphic density-like ideals that are not generalized density ideals (in this case, I contributed by indicating in [H7, Theorem 4.22] a general construction of tall density-like ideals that are not generalized density ideals).

Moreover, we have given a characterization of generalized density ideals, which resembles the definition of density-like ideals ([H7, Theorem 5.3]).

Katětov's order in the article [H7] is used in the proof of item (a). Firstly, in [H7, Theorem 4.3 and Theorem 4.4] we showed that if $\mathcal{I}$ is a tall density-like ideal (in particular, a tall generalized density ideal), then $(\{\emptyset\} \otimes \mathrm{Fin}) \cap (\mathcal{I} \otimes \{\emptyset\})$ is a density-like ideal which is not a generalized density ideal. We then realized that if $\mathcal{I}$ and $\mathcal{J}$ are incomparable in the Katětov order, then $(\{\emptyset\} \otimes \mathrm{Fin}) \cap (\mathcal{I} \otimes \{\emptyset\})$ and $(\{\emptyset\} \otimes \mathrm{Fin}) \cap (\mathcal{J} \otimes \{\emptyset\})$ are not isomorphic. Thus, we could take an antichain $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ in $\leq_K$, composed of tall generalized density ideals (the existence of such an antichain was known before), and conclude that the ideals $(\{\emptyset\} \otimes \mathrm{Fin}) \cap (\mathcal{I}_\alpha \otimes \{\emptyset\})$, for $\alpha < \mathfrak{c}$, are as needed in item (a).

4.2 Scientific achievements not included in the habilitation series

I will now proceed to a brief discussion of my scientific achievements not included in the habilitation series, which consists of 20 scientific articles (in the order they will be discussed below):

[O1] Adam Kwela and Ireneusz Recław, *Ranks of $\mathcal{F}$-limits of filter sequences*, J. Math. Anal. Appl. **398** (2013), no. 2, 872–878. MR 2990109

[O2] Adam Kwela and Marcin Sabok, *Topological representations*, J. Math. Anal. Appl. **422** (2015), no. 2, 1434–1446. MR 3269521

[O3] Adam Kwela, *A note on a new ideal*, J. Math. Anal. Appl. **430** (2015), no. 2, 932–949. MR 3351990

[O4] Adam Kwela and Piotr Zakrzewski, *Combinatorics of ideals – selectivity versus density*, Comment. Math. Univ. Carolin. **58** (2017), no. 2, 261–266. MR 3666945

[O5] Adam Kwela, *Erdős-Ulam ideals vs. simple density ideals*, J. Math. Anal. Appl. **462** (2018), no. 1, 114–130. MR 3771234

[O6] Adam Kwela, Michał Popławski, Jarosław Swaczyna, and Jacek Tryba, *Properties of simple density ideals*, J. Math. Anal. Appl. **477** (2019), no. 1, 551–575. MR 3950052

[O7] Kumardipta Bose, Pratulananda Das, and Adam Kwela, *Generating new ideals using weighted density via modulus functions*, Indag. Math. (N.S.) **29** (2018), no. 5, 1196–1209. MR 3853420

[O8] Adam Kwela and Jacek Tryba, *Homogeneous ideals on countable sets*, Acta Math. Hungar. **151** (2017), no. 1, 139–161. MR 3594409

[O9] Adam Kwela and Marcin Staniszewski, *Ideal equal Baire classes*, J. Math. Anal. Appl. **451** (2017), no. 2, 1133–1153. MR 3624783

[O10] Adam Kwela, *Ideal weak QN-spaces*, Topology Appl. **240** (2018), 98–115. MR 3784399

[O11] Adam Kwela, *Inductive limits of ideals*, Topology Appl. **300** (2021), Paper No. 107798, 13. MR 4293085

[O12] Rafał Filipów, Adam Kwela, and Jacek Tryba, *The ideal test for the divergence of a series*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 3, 98.

[O13] Adam Kwela, *Additivity of the ideal of microscopic sets*, Topology Appl. **204** (2016), 51–62. MR 3482702

[O14] Klaudiusz Czudek, Adam Kwela, Nikodem Mrożek, and Wojciech Wołoszyn, *Ideal-like properties of generalized microscopic sets*, Acta Math. Hungar. **150** (2016), no. 2, 269–285. MR 3568089

[O15] Adam Kwela, *Haar-smallest sets*, Topology Appl. **267** (2019), 106892, 25. MR 4008566

[O16] Adam Kwela and Wojciech Wołoszyn, *Differentiability of continuous functions in terms of Haar-smallness*, Topology Appl. **284** (2020), 107353, 16. MR 4138435

[O17] Paweł Klinga and Adam Kwela, *Borel complexity of the family of attractors for weak IFSs*, Acta Math. Hungar. **166** (2022), no. 1, 124–141. MR 4385979

[O18] Paweł Klinga, Adam Kwela, and Marcin Staniszewski, *Size of the set of attractors for iterated function systems*, Chaos Solitons Fractals **128** (2019), 104–107. MR 3985600

[O19] Paweł Klinga and Adam Kwela, *Porosities of the sets of attractors*, J. Math. Anal. Appl. **514** (2022), no. 2, Paper No. 126348, 10. MR 4429586

[O20] Paweł Klinga and Adam Kwela, *Comparison of the sets of attractors for systems of contractions and weak contractions*, Chaos Solitons Fractals **155** (2022), Paper No. 111764, 6. MR 4362797

For greater clarity of the description, these articles are divided into six groups:

1. Results obtained before the PhD: [O1], [O2], [O3] and [O4].
2. Study of certain classes of analytic P-ideals: [O5], [O6] and [O7].
3. Other results concerning ideals: [O8], [O9], [O10], [O11] and [O12].
4. Study of microscopic sets: [O13] and [O14].
5. Study of Haar-small sets: [O15] and [O16].
6. Study of attractors of iterated function systems: [O17], [O18], [O19] and [O20].

4.2.1 Results obtained before the PhD

Some of the articles discussed in this section were published after I had been awarded the PhD degree, however, this is only the result of a long editorial process. My PhD dissertation was based on the articles [O1], [O2] and [O3], while the paper [O4] was not included in it.

In [O1] we showed that if $\mathcal{I}$ is a Borel ideal of rank 1, then $\mathrm{rk}(\mathcal{I}\otimes\mathcal{J}) = \mathrm{rk}(\mathcal{J})+1$ for any analytic ideal $\mathcal{J}$ ([O1, Corollary 3.6]) – this is a strengthening of [E33, Proposition 4.4], where the exact rank of $\mathcal{I}\otimes\mathcal{J}$ is obtained only in the case of $\mathcal{I}=\mathrm{Fin}$. Our proof used the game $G'(\mathcal{I})$ described in subsections 4.1.4 and 4.1.6. Moreover, we proved that for every ordinal $0<\alpha<\omega_1$ there exist Borel ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\alpha = \mathrm{rk}(\mathcal{I}) = \mathrm{rk}\ (\mathcal{J})$, but $\mathrm{rk}(\mathcal{I}\cap\mathcal{J}) = 1$ ([O1, Lemma 4.4]).

We will say that an ideal $\mathcal{I}$ is topologically representable if there is a separable metrizable space X, a countable dense set $D\subseteq X$ and a σ-ideal I on X containing all singletons, such that $\mathcal{I}$ is isomorphic to the ideal $\left\{A\subseteq D:\ \overline{A}\in I\right\}$ (here $\overline{A}$ is the closure of A in X; studies of similar forms of representability of ideals are also conducted by P. Borodulin-Nadzieja, B. Farkas and G. Plebanek – see [E15] and [E16]). The starting point for the paper [O2] was the conjecture formulated by M. Sabok and J. Zapletal ([E82, Conjecture 4.4]): an analytic ideal is topologically representable if and only if it is a tall, weakly selective ideal of class $\mathbf{\Pi^0_3}$ ($\mathcal{I}$ is weakly selective if every function defined on some $\mathcal{I}$-positive set is either constant or one-to-one on some $\mathcal{I}$-positive subset). An additional motivation for our research were applications of topologically representable ideals in forcing theory (see [E82]) and in the theory of equivalence relations on 2^ω (see [E94]).

The main result of [O2] is the characterization of topologically representable ideals ([O2, Theorem 1.1]), which does not solve [E82, Conjecture 4.4], however, unlike its formulation, it is purely combinatorial. Furthermore, we have shown that all topologically representable ideals are Rudin-Blass equivalent[9] ([O2, Corollary 1.2]) and that every topologically representable analytic ideal is $\mathbf{\Pi^0_3}$-complete ([O2, Theorem 1.4]). Moreover, we have characterized weakly selective coanalytic ideals with the use of topologically representable ideals ([O2, Theorem 1.6]), applying in the proof a certain ideal game considered earlier by C. Laflamme in [E71] and by M. Hrušák in [E54].

We say that an ideal $\mathcal{I}$ on ω is weakly Ramsey if every tree $T\subseteq\omega^{<\omega}$ satisfying for each $s\in T$ the condition $\{n\in\omega:\ s^\frown(n)\in T\}\in\mathcal{I}^\star$, has a branch in $\mathcal{I}^+$. This property was introduced by C. Laflamme in [E71] to characterize when one of the players has a winning strategy in the game from the previous paragraph. In [O3] I defined the ideal $\mathcal{WR}$ of class $\mathbf{\Sigma^0_2}$ with the following property: an ideal $\mathcal{I}$ is weakly Ramsey if and only if $\mathcal{WR}\not\leq_K\mathcal{I}$ ([O3, Theorem 1.3]). Using the earlier result of T. Natkaniec and P. Szuca from [E77], this means that $\mathcal{WR}$ is a critical ideal for pointwise convergence with respect to an ideal of sequences of quasi-continuous functions[10] (in the same sense that Fin^2 is critical for continuous functions – see subsection 4.1.7). I also showed that $\mathcal{WR}$ is an example of the ideal that R. Filipów, N. Mrożek, I. Recław and P. Szuca asked about in [E42, discussion before Theorem 3.11]. Research on Ramsey properties of ideals was later continued by M. Hrušák, D. Meza-Alcántara, E. Thümmel and C. Uzcátegui in [E56].

An ideal $\mathcal{I}$ is selective if for each partition $(X_n)_{n\in\omega}$ of the set $\bigcup\mathcal{I}$, either $\bigcup_{n\leq k}X_n\in\mathcal{I}^\star$ for some $k\in\omega$, or one can find a selector of that partition belonging to $\mathcal{I}^+$. The classic result of A. R. D. Mathias is that an analytic or coanalytic selective ideal cannot be tall ([E73]). In [O4] we examined how far apart the concepts of selectivity and tallness are.

4.2.2 Study of certain classes of analytic P-ideals

We will call an ideal a simple density ideal if it is of the form:

$$\mathcal{Z}_g = \left\{A\subseteq\omega:\ \lim_{n\to\infty}\frac{|A\cap n|}{g(n)}=0\right\},$$

where $g:\omega\to(0,\infty)$ is such that $\lim_{n\to\infty}g(n)=\infty$ and $\frac{n}{g(n)}$ does not converge to 0. This class of ideals was introduced in [E5], where it was shown, among others, that there is no inclusion between the class of simple density ideals and the class of Erdős-Ulam ideals (an ideal is Erdős-Ulam if it is of the form:

$$\mathcal{EU}_h = \left\{A\subseteq\omega:\ \lim_{n\to\infty}\frac{\sum_{i\in A\cap n}h(i)}{\sum_{i\in n}h(i)}=0\right\},$$

where $h:\omega\to[0,\infty)$ is such that $\sum_{i\in\omega}h(i)=\infty$; these ideals were introduced in [E61] by W. Just and A. Krawczyk in the context of the question posed by P. Erdős and S. Ulam, concerning isomorphisms of Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$.

[9] For the definition of Rudin-Blass order, see [O2].

[10] The definition of quasi-continuity can be found in [O3] or in [E77].

Recently, J. Tryba in [E90] showed that the often used characterization of Erdős-Ulam ideals ([E37, Theorem 1.13.3]) actually characterizes only ideals isomorphic to Erdős-Ulam ideals, which necessitates a slight reformulation of some of the results listed below. My achievements in this field include, among others:

- [O5, Theorem 2]: A simple density ideal is isomorphic to an Erdős-Ulam ideal if and only if it contains the ideal $\mathcal{I}_d$.
- [O5, Theorem 3]: Characterization when an ideal isomorphic to an Erdős-Ulam ideal is a simple density ideal.
- [O6, Theorem 3]: There is an antichain in the Katětov order of cardinality $\mathfrak{c}$ composed of simple density ideals – this is a strengthening of [E5, Theorem 2.7], where instead of $\leq_K$ the inclusion is considered.
- [O5, Proposition 5]: There is a family of cardinality $\mathfrak{c}$ composed of pairwise non-isomorphic Erdős-Ulam ideals that are not simple density ideals – all Erdős-Ulam ideals are equivalent in the sense of $\leq_K$ (by [E37, Theorem 1.13.10]), so in this case it is not possible to construct an antichain in the sense of Katětov order.
- [O6, Theorem 6]: If $\mathcal{Z}_g \cong \mathcal{Z}_g \restriction A$ for some $A \subseteq \omega$, then the increasing enumeration of A witnesses this isomorphism – this is a partial answer to [O8, Problem 5.8], where together with J. Tryba we asked about the characterization of ideals with the above property.

Moreover, in [O7] we investigated some further generalization of simple density ideals.

4.2.3 Other results concerning ideals

In the article [O8] we studied homogeneous ideals, i.e. such ideals $\mathcal{I}$ that $\mathcal{I} \cong \mathcal{I} \restriction A$ for all $A \in \mathcal{I}^+$. We obtained their convenient characterization ([O8, Corollary 2.2]), which allowed us to show that, for example, $\mathcal{W}$ and Fin^n, for all $n \in \omega \setminus \{0\}$, are homogeneous. As it turned out after the publication of [O8], the aforementioned characterization is a positive answer to [E75, Question 2.1.10] posed by D. Meza-Alcántara, while [O8, Example 2.6], in which we show that $\mathcal{W}$ is homogeneous, answers [E55, Question 5.11] posed by M. Hrušák. Moreover, in [O8] we answer a number of questions posed in [E6] by M. Balcerzak, S. Głąb and J. Swaczyna.

In [O9] we studied families of functions of first Baire class for quasi-normal convergence with respect to a given ideal (more precisely: the family of real-valued functions defined on a perfectly normal space, being a quasi-normal limit with respect to a given ideal of a sequence of continuous functions; quasi-normal convergence with respect to a given ideal is described in subsection 4.1.6). In particular, we characterized for which Borel ideals this family coincides with the classic family of functions of first Baire class (using in the proof the technique described in subsection 4.1.6 and based on the game $G'(\mathcal{I})$). We also examined quasi-normal convergence with respect to a fixed ideal of sequences of quasi-continuous functions (using my previous results from [O3]). Our article was motivated by the papers [E33] and [E70], in which similar studies were carried out in the case of pointwise convergence with respect to an ideal of sequences of continuous functions, paper [E45], which contains considerations similar to ours, as well as the paper [E77], in which the pointwise convergence with respect to an ideal of sequences of quasi-continuous functions was investigated.

In the article [O10] I studied $\mathcal{I}$wQN-spaces[11] (i.e., for a given ideal $\mathcal{I}$, topological spaces X such that for any pointwise convergent to 0 sequence of continuous functions $(f_n)_{n\in\omega} \in \left(\mathbb{R}^X\right)^\omega$ one can find its subsequence convergent to 0 quasi-normally with respect to $\mathcal{I}$; see subsection 4.1.6 for similar spaces – there, however, we consider sequences of continuous functions converging with respect to $\mathcal{I}$ instead of Fin, and we require that the entire sequence $(f_n)_{n\in\omega}$ converges quasi-normally with respect to $\mathcal{I}$). These spaces, similarly to $\mathcal{I}$QN-spaces, are thoroughly studied in Košice, e.g. by L. Bukovský, M. Repický and J. Šupina (see [E20], [E80], [E86] and [E89]). I characterized combinatorially, for any ideal $\mathcal{I}$, the minimal cardinality of a non-$\mathcal{I}$wQN-space ([O10, Theorem 2.3]) and showed that it is equal to $\mathfrak{b}$ for all $\mathbf{\Sigma^0_2}$ ideals ([O10, Theorem 2.7]). I also proved that consistently there is an ideal $\mathcal{I}$ such that $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$ (by [E89, Theorem 1.4],

[11]The famous Scheepers' conjecture concerns a characterization of FinwQN-spaces – see [E89].

if $\mathrm{Fin}^2 \leq_K \mathcal{I}$, then every topological space is an $\mathcal{I}$wQN-space) and an $\mathcal{I}$wQN-space that is not a FinwQN-space ([O10, Theorem 2.11]) – this is a partial answer to [E20, Problem 3.7]. I also showed that if $\mathcal{I}$ is a tall ideal and

$$\mathtt{cov}^\star(\mathcal{I}) = \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \mathcal{I}\ \wedge\ \forall_{S\in[\omega]^\omega}\ \exists_{A\in\mathcal{A}}\ |A \cap S| = \omega\right\},$$

then any topological space X of cardinality less than $\mathtt{cov}^\star(\mathcal{I})$ is an $\mathcal{I}$wQN-space if and only if it is a FinwQN-space – this answered [E20, Problem 3.2] and inspired some results from [E86]. It is worth noting that $\mathtt{cov}^\star(\mathcal{I})$ is a well-studied object and can have relatively large values, e.g. $\mathtt{cov}^\star(\mathrm{conv}) = \mathfrak{c}$ (see [E54]).

In [O11] I showed that the conjecture of G. Debs and J. Saint Raymond from [E33], described in subsection 4.1.7, is false for $\alpha = \omega$ (the main result of [H6] says that it is false for $\alpha = 3$). The ideal Fin'_ω, which is a suitable counterexample, has another interesting property: $\mathrm{Fin}'_\omega \leq_K \mathcal{I}$ if and only if $\mathrm{Fin}^n \leq_K \mathcal{I}$ for all $n \in \omega \setminus \{0\}$.

The article [O12] deals with generalizations of classic Olivier's theorem saying that for any non-increasing sequence $(a_n)_{n\in\omega}$ of positive real numbers, if the series $\sum_{n\in\omega} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. We have proved general theorems from which many of the known generalizations of Olivier's theorem can be easily derived, as well as obtained some new conclusions. In all cases, the convergence of $(na_n)_{n\in\omega}$ is replaced by ideal convergence. In the second part of the article, we investigated the sizes of families of sequences for which our theorems fail – we were interested in finding large linear and algebraic substructures in these families. The main inspiration for this part was the paper [E12] written by A. Bartoszewicz, S. Głąb and A. Widz.

4.2.4 Study of microscopic sets

A set $A \subseteq \mathbb{R}$ is called microscopic ($A \in \mathcal{M}icro$) if for each $\varepsilon > 0$ there is a sequence of intervals $(I_n)_{n\in\omega\setminus\{0\}}$ such that $A \subseteq \bigcup_{n\in\omega\setminus\{0\}} I_n$ and $\mathrm{lh}(I_n) \leq \varepsilon^n$ for each $n \in \omega \setminus \{0\}$ (where $\mathrm{lh}(I)$ denotes the length of the interval I). $\mathcal{M}icro$ is a σ-ideal contained in the σ-ideal of sets of Lebesgue measure zero (and even in the σ-ideal of sets of Hausdorff dimension zero). On the other hand, the ternary Cantor set on the real line is of Lebesgue measure zero but not in $\mathcal{M}icro$ (for an example of a set of Hausdorff dimension zero not in $\mathcal{M}icro$ see [E2]). Microscopic sets were introduced in [E3], and were later studied, among others, in [E2] and [E26], and by a number of mathematicians from Łódź (see [E53]). Studies of these sets are often inspired by and related to studies of strong measure zero sets (any strong measure zero set is microscopic, while an example showing that the opposite inclusion does not hold can be found in [E40]).

My main results on this topic include:

- [O13, Theorem 3.2]: The additivity of $\mathcal{M}icro$ is equal to ω_1 (in ZFC), i.e. there is a family of microscopic sets of cardinality ω_1 whose union is not a microscopic set – it is a surprising answer (previously it was supposed that, similarly to the σ-ideal of strong measure zero sets, the additivity of $\mathcal{M}icro$ should be equal to $\mathfrak{c}$, assuming Martin's axiom) to the question posed by G. Horbaczewska during her talk at the XXIV Summer Conference on Real Functions Theory (Stara Leśna, 2010).

- [O13, Proposition 4.5]: Some modification of $\mathcal{M}icro$ (obtained by replacing $\mathrm{lh}(I_n) \leq \varepsilon^n$ in the definition of $\mathcal{M}icro$ with $\mathrm{lh}(I_n) \leq \varepsilon^{\ln(n+2)}$) has additivity $\mathfrak{c}$, assuming Martin's axiom.

- [O14, Theorem 3.6]: The family of nanoscopic sets (created by replacing $\mathrm{lh}(I_n) \leq \varepsilon^n$ with $\mathrm{lh}(I_n) \leq \varepsilon^{2^n}$ in the definition of $\mathcal{M}icro$) is not an ideal (more precisely: there are two nanoscopic Borel sets whose union is not nanoscopic) – this answers the question posed by G. Horbaczewska in [E51].

Research on microscopic sets and their generalizations is being continued in Łódź (see [E52], [E62], [E63], [E78] and [E79]).

4.2.5 Study of Haar-small sets

If $\mathcal{F} \subseteq \mathcal{P}(2^\omega)$ is a family of sets closed under taking subsets of its elements, then we say that a subset A of an abelian Polish group X is Haar-$\mathcal{F}$ if there is a Borel set $B \supseteq A$ and a continuous function $f : 2^\omega \to X$, such that $f^{-1}[B + x] \in \mathcal{F}$ for all $x \in X$.

It was shown in [E8] that if $\mathcal{F}$ is the σ-ideal of Lebesgue measure zero subsets of 2^ω (meager subsets of 2^ω), the notion of Haar-$\mathcal{F}$ sets coincides with the well-known notion of Haar-null (Haar-meager) sets, introduced in [E25] and [E58] (in [E30], respectively). Also, it is easy to see that if $\mathcal{F} \subseteq \mathcal{F}'$, then every Haar-$\mathcal{F}$ set is also Haar-$\mathcal{F}'$.

In my work, I studied Haar-$\mathcal{F}$ sets for the following families $\mathcal{F}$: $[2^\omega]^{<\omega_1}$, $[2^\omega]^{<\omega}$, $[2^\omega]^{\leq n}$ for $n \in \omega$ (concepts closely related to Haar-$\mathcal{F}$ sets for the above families have previously been studied also by U. B. Darji and T. Keleti in [E31], M. Elekes and J. Steprāns in [E36], M. Balcerzak in [E4], P. Zakrzewski in [E93], and T. Banakh, N. Lyaskovska and D. Repovš in [E10]). My main results on this topic include:

- [O15, Proposition 3.1]: All countable subsets of an abelian Polish group are Haar-$[2^\omega]^{\leq 1}$.
- [O15, Theorem 4.1]: Cantor ternary set on the real line is Haar-$[2^\omega]^{\leq 2}$, but not Haar-$[2^\omega]^{\leq 1}$.
- [O16, Theorem 2.2]: The set $\mathcal{SD}[0,1]$ of real functions defined on $[0,1]$ that are differentiable in at least one point, is not Haar- $[2^\omega]^{<\omega_1}$ in the space $C[0,1]$ – this result is somewhat contrary to the series of results saying that $\mathcal{SD}[0,1]$ is a small set in the space $C[0,1]$ (e.g. in the classic paper [E7], S. Banach proved that $\mathcal{SD}[0,1]$ is meager in $C[0,1]$, while in [E57] B. Hunt showed that $\mathcal{SD}[0,1]$ is Haar-null in $C[0,1]$).
- [O15, Theorem 6.1]: The family of Haar-$[2^\omega]^{<\omega}$ subsets of $\mathbb{R}$ is not an ideal (more precisely: there are two compact Haar-$[2^\omega]^{<\omega}$ subsets of $\mathbb{R}$ whose union is not Haar-$[2^\omega]^{<\omega}$) – this answers the question posed by T. Banakh, S. Głąb, E. Jabłońska and J. Swaczyna in the first versions of [E8] (available on ArXiv[12]), and by J. Swaczyna during his talk at the XLI Summer Symposium in Real Analysis conference (Wooster, 2017), while a slight modification of this result (see [O15, Corollary 6.2]) answers the question posed in the first version of [E9] (available on ArXiv[13]), and by T. Banakh during his talk at the Frontiers of Selection Principles conference (Warsaw, 2017).

In addition, I distinguished the families Haar-$[2^\omega]^{<\omega_1}$, Haar-$[2^\omega]^{<\omega}$ and Haar-$[2^\omega]^{\leq n}$, for $n \in \omega$, in any space of the form $\mathbb{R} \times X$, where X is an abelian Polish group ([O15, Proposition 2.5, Theorem 5.1, Corollary 6.3 and Theorem 7.1]).

4.2.6 Study of attractors of iterated function systems

Recall that a function $f : X \to X$, where (X,d) is a compact metric space, is called a:

- contraction if there is a constant $c \in [0,1)$ such that $d(f(x),f(y)) \leq c \cdot d(x,y)$ for all $x, y \in X$, $x \neq y$;
- weak contraction if $d(f(x),f(y)) < d(x,y)$ for all $x, y \in X$, $x \neq y$.

A compact set $A \subseteq X$ is an attractor of an iterated function system ($A \in \mathcal{A}(X)$) if there are $n \in \omega$ and contractions $f_0, \ldots, f_n : X \to X$ such that $A = \bigcup_{i \leq n} f_i[A]$.[14] If in the above definition we replace contractions with weak contractions, then we get an attractor of a weak iterated function system[15] and write: $A \in \mathcal{A}_w(X)$. Clearly, $\mathcal{A}(X) \subseteq \mathcal{A}_w(X)$ in every compact metric space X.

In my works I studied the properties of $\mathcal{A}(X)$ and $\mathcal{A}_w(X)$ as subsets of the space $\mathcal{K}(X)$ of all compact non-empty subsets of X with the Hausdorff metric. The starting point was the result that $\mathcal{A}([0,1]^d)$ is a meager $\mathbf{\Sigma^0_2}$ set, for all $d \in \omega \setminus \{0\}$ (see [E27], [E28] and [E87]). The following results of mine on this topic are true for all $d \in \omega \setminus \{0\}$:

- [O17, Theorem 3.1 and Theorem 3.5]: $\mathcal{A}_w([0,1])$ is an analytic set, but not of class $\mathbf{\Sigma^0_2}$.

[12] The work [E8] underwent a very long editorial process, during which numerous changes were made. I have obtained my result at the beginning of this process, and managed to publish it before the final version of [E8] was released. Since the authors knew about my answer to their question, they decided not to include this question in the published version of the article, and to cite my result instead.

[13] The authors of [E9] knew about my answer to their question before the final version of their article was published, so [E9] cites my result instead of posing the question.

[14] Each iterated function system, i.e. a sequence of contractions $f_0, \ldots, f_n : X \to X$, has a unique attractor (by Banach's contraction theorem applied to the space $\mathcal{K}(X)$ of all compact non-empty subsets of X and the so-called Hutchinson operator $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ given by $\mathcal{F}(A) = \bigcup_{i \leq n} f_i[A]$).

[15] In this case, also, each sequence of weak contractions has a unique attractor – see [E35].

- [O18, Theorem 4.1]: $\mathcal{A}_w([0,1]^d)$ is a meager set – this is a strengthening of the fact that $\mathcal{A}([0,1]^d)$ is meager; this result was also obtained independently by E. D'Aniello and T. H. Steele in [E29, Theorem 4.1], and then generalized to all perfect Polish spaces by K. Leśniak, N. Snigireva and F. Strobin in [E72, Theorem 4.2].
- [O18, Theorem 3.1]: $\mathcal{A}([0,1]^d)$ is σ-porous[16] – this is another strengthening of the fact that $\mathcal{A}([0,1]^d)$ is meager.
- [O19, Theorem 4.1]: $\mathcal{A}([0,1])$ is not σ-strongly porous.[17]
- [O20, Theorem 3.4]: There is a set $A \in \mathcal{K}([0,1]^d) \setminus A_w([0,1]^d)$ which is a limit of some sequence of attractors of iterated function systems composed of only two contractions.

5 Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

Since January 2015, I am employed at the Institute of Mathematics of the University of Gdańsk (UG), while my PhD studies have been completed at the Institute of Mathematics of the Polish Academy of Sciences (IMPAN). In addition, during my PhD studies, I have completed two scientific internships at the Faculty of Mathematics, Computer Science and Mechanics of the University of Warsaw (UW):

1. Internship within SSDNM PhD studies programme: 1 academic semester (February – May 2012).
2. Internship for PhD students awarded by the Warsaw Center of Mathematics and Computer Science: 1 academic semester (October 2013 – January 2014).

During the period in which I have been associated with IMPAN, I obtained the results contained in four scientific articles: [O1] written jointly with Prof. I. Recław from UG, [O2] written jointly with Dr M. Sabok (who later became the auxiliary supervisor of my PhD) from IMPAN, [O3] written alone and [O4] written jointly with Prof. P. Zakrzewski (who was the main supervisor of my PhD) from UW. In the articles [O1] and [O2] I have IMPAN affiliation, and in [O3] and [O4] I have UG affiliation, because I finished working on them when I was already employed at UG.

All my results obtained before my PhD concerned combinatorial and descriptive properties of ideals on countable sets, i.e. issues closely related to the subject of the habilitation series. For example, in the article [O2] (the main results of which were obtained during the first of the above internships) we used a certain game associated to an ideal, similar to those used in the habilitation series. In turn, in the paper [O3] (main results of which were obtained during the second of the above-mentioned internships – relevant information can be found on the title page of the article) I obtained my first results regarding the Katětov order – I characterized the so-called weakly Ramsey ideals (this is described in more detail in subsection 4.2.1). The habilitation series is a far-reaching development of this method of examining ideals on countable sets. The experience and knowledge gained during this period contributed significantly to the habilitation series.

During my PhD studies, I have received a grant from the National Science Center (awarded in the Preludium competition). I carried out this grant at IMPAN in the period from August 19, 2013 to February 18, 2016 (so also after my PhD, while being already employed at UG). The grant resulted in 4 publications: [O2], [O3] and [O4] written before my PhD, and [O9] written after my PhD jointly with M. Staniszewski from UG (I have acted as an auxiliary supervisor

[16] The definition of σ-porosity comes from E. P. Dolženko ([E34]). Every σ-porous set is meager, but the opposite implication does not hold. Many papers aiming to use σ-porosity to strengthen the classic results saying that some set is meager have been published (e.g. the classic result of S. Banach from [E7], mentioned in susbection 4.2.5 and saying that the set $\mathcal{SD}[0,1]$ is meager in $C[0,1]$ was strengthened by V. Anisiu in [E1] and by P. M. Gandini and A. Zucco in [E49]: $\mathcal{SD}[0,1]$ is also σ-porous in $C[0,1]$).

[17] For the definition of σ-strong porosity see [O19] or [E74]. More about comparing different types of porosity in abstract spaces can be found in [E92].

of his PhD and the paper [O9] became part of his dissertation) – relevant information can be found on the title pages of these articles. In the last of these papers I used the ideal game $G'(\mathcal{I})$ described in subsections 4.1.4 and 4.1.6.

6 Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

6.1 Teaching achievements

Conducted academic courses after PhD:

1. Course *Data analysis laboratory* for *Mathematical Modeling and Data Analysis* students (2nd year), University of Gdańsk, lectures (30 hours) in academic years 2017/2018–2022/2023, laboratory tutorials (30 hours) in academic years 2017/2018–2020/2021.
2. Course *Statistical inference 1* for *Mathematical Modeling and Data Analysis* students (2nd year), University of Gdańsk, lectures (30 hours) in academic year 2022/2023.
3. Course *Mathematical analysis 1* for *Mathematical Modeling and Data Analysis* students (1st year), University of Gdańsk, tutorials (60 hours) in academic years 2020/2021–2022/2023.
4. Course *Mathematical analysis 2* for *Mathematical Modeling and Data Analysis* students (1st year), University of Gdańsk, tutorials (60 hours) in academic years 2020/2021 and 2021/2022.
5. Course *Mathematical analysis 3* for *Mathematical Modeling and Data Analysis* students (2nd year), University of Gdańsk, tutorials (60 hours) in academic years 2021/2022 and 2022/2023.
6. Course *Descriptive statistics* for *Mathematical Modeling and Data Analysis* students (1st year), University of Gdańsk, laboratory tutorials (30 hours) in academic years 2018/2019, 2019/2020 and 2021/2022.
7. Course *Data analysis* for *Mathematics* students (3rd year), University of Gdańsk, lectures (30 hours) in academic years 2015/2016–2020/2021, laboratory tutorials (30 hours) in academic years 2015/2016–2020/2021.
8. Course *Mathematics* for *Chemistry* students (1st year), University of Gdańsk, tutorials (60 hours) in academic year 2020/2021.
9. Course *Mathematical software* for *Mathematics* students (1st year), University of Gdańsk, laboratory tutorials (30 hours) in academic years 2015/2016–2019/2020.
10. Course *Probability Theory* for *Informatics* students (2nd year), University of Gdańsk, tutorials (60 hours) in academic year 2018/2019.
11. Course *Mathematics 1* for *Biotechnology* students (1st year), University of Gdańsk, tutorials (45 hours) in academic year 2017/2018.
12. Course *Introduction to set theory and logic* for *Mathematics* students (1st year), University of Gdańsk, tutorials (30 hours) in academic year 2016/2017.
13. Course *Mathematical Analysis 2* for *Mathematics* students (1st year), University of Gdańsk, tutorials (60 hours) in academic year 2014/2015.
14. Course *Mathematical Analysis 1* for *Physics* students (1st year), University of Gdańsk, tutorials (60 hours) in academic year 2014/2015.
15. Course *Mathematical Analysis 2* for *Physics* students (1st year), University of Gdańsk, tutorials (60 hours) in academic year 2014/2015.
16. Course *Basics of programming* for *Mathematics* students (1st year), University of Gdańsk, laboratory tutorials (30 hours) in academic year 2014/2015.

Conducted academic courses before PhD:

1. Course *Linear algebra* for *Economics* students (1st year), University of Warsaw (as part of Warsaw Center of Mathematics and Computer Science PhD internship), tutorials (60 hours) in academic year 2013/2014.
2. Course *Stochastic processes* for *Mathematics* students (3rd year), University of Gdańsk, tutorials (30 hours) in academic year 2011/2012.
3. Course *Mathematics* for *Finance and Accounting* students (1st year), University of Gdańsk, tutorials (25 hours) in academic year 2010/2011.

Acting as an auxiliary supervisor in completed PhD procedures:

1. *Name*: Jacek Tryba
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
   Title of the PhD dissertation: Analytical properties of ideals
   Principal supervisor: Prof. UG, Dr hab. Rafał Filipów
   Date of the award of PhD: June 2018
2. *Name*: Marcin Staniszewski
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
   Title of the PhD dissertation: Ideal e-convergence of sequences of functions
   Principal supervisor: Dr hab. Rafał Filipów
   Date of the award of PhD: November 2016

Acting as an auxiliary supervisor in ongoing PhD procedures:

1. *Name*: Krzysztof Kowitz
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
   Title of the PhD dissertation: Applications of Katětov order in studies of topological spaces and ultrafilters
   Principal supervisor: Prof. UG, Dr hab. Rafał Filipów
   Expected year of the award of PhD: 2023

Supervision of master's theses:

1. *Name of the student*: Wojciech Wołoszyn
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
   Year of the master's degree: 2020
2. *Name of the student*: Agnieszka Brzostek
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk
   Year of the master's degree: 2019

6.2 Achievements in popularization of science

- Lecture "Mathematics is needed" for Primary School No. 9 in Rumia, Poland, 2016.
- Help in organizing the event "The Great Final of the 1st edition of Pomeranian Mathematical Matches" – hosting one of the matches, April 2016.
- Help in organizing the event "Mathematics as an important component of culture and civilization – Mathematical Mikołajki" at the University of Gdańsk, December 2016.

6.3 Organizational achievements

Organization of scientific conferences:

1. *Name of the conference*: Gdańsk Logic Conference
   Dates: 5–7 May 2023
   Institution, place: University of Gdańsk, Gdańsk, Poland
   Role: Organizing Committee member

2. *Name of the conference*: Workshop on Set Theory and its Applications to Topology and Real Analysis, in memory of Irek Recław
Dates: 4–6 July 2013
Institution, place: University of Gdańsk, Gdańsk, Poland
Role: Organizing Committee member

Other organizational activities:

- In 2022, I was the chairman, and in 2021 – the secretary, of the recruitment committee for the *Mathematics* and *Mathematical Modeling and Data Analysis* studies at University of Gdańsk.
- In years 2017–2019, I was a member of the Council of the Faculty of Mathematics, Physics and Informatics and the Council of the Institute of Mathematics at the University of Gdańsk.

7 Apart from information set out in 1-6 above, the applicant may include other information about his/her professional career, which he/she deems important

7.1 Scientometrics information (as of May 8, 2023)

Citations:

- Web of Science: 95 citations (46 without self-citations);
  53 citing articles (34 without self-citations)
- MathSciNet: 86 citations (by 48 authors)
- Google Scholar: 151 citations (119 since 2018)

Hirsch index:

- Web of Science: 6
- Google Scholar: 7 (7 since 2018)

Total points awarded by the Ministry of Science and Higher Education:

- since 2020 (i.e. since the change to the scale of up to 200 points):
  1610 (16 scientific articles)
- before 2019 (i.e. to the change from the scale of up to 50 points to the scale of up to 200 points):
  280 (11 scientific articles), including 110 before the award of the PhD degree (4 scientific articles) and 170 after the award of the PhD degree (7 articles)

Total Impact Factor: 34,303 (27 scientific articles)

7.2 Awards and grants

- Team Award of the 2nd degree of the Rector of the University of Gdańsk for scientific achievements documented by publications in year 2021.
- Research grant for PhD students, founded by the National Science Centre (Poland) in the Preludium contest
  Title of the project: Combinatorial and descriptive properties of ideals on countable sets
  Amount: 62 400 PLN
  Start/end dates: August 19, 2013 – February 18, 2016
  Role: Applicant, PI

- Research grants awarded to young scientists, founded by the University of Gdańsk:
    1. *Title of the project*: Generalizations of the notion of density of subsets of the set of natural numbers
    Amount: 8 000 PLN
    Year: 2017
    Role: Applicant, PI
    2. *Title of the project*: Applications of ideal convergence
    Amount: 6 000 PLN
    Year: 2016
    Role: Applicant, PI
    3. *Title of the project*: Different kinds of ideal convergence
    Amount: 4 000 PLN
    Year: 2015
    Role: Applicant, PI

7.3 Presentations on conferences and seminars

During my career I have delivered:

- 2 presentations on international scientific conferences upon invitation:
    - Set Theory Workshop in Vienna, planned on 12.06.2023, Vienna, Austria,
    - Frontiers of Selection Principles, 20.08–1.09.2017, Warsaw, Poland,
- 10 presentations on international scientific conferences (contributed):
    - European Set Theory Conference, 29.08–2.09.2022, Turin, Italy,
    - Set-theoretic methods in topology and real functions theory, 9–13.09.2019, Košice, Slovakia,
    - Winter School in Abstract Analysis, Section: Set Theory and Topology, 27.01–3.02.2018, Hejnice, Czech Republic,
    - Winter School in Abstract Analysis, Section: Set Theory and Topology, 28.01–4.02.2017, Hejnice, Czech Republic,
    - SETTOP 2016, 20–23.06.2016, Novi Sad, Serbia,
    - European Set Theory Conference, 24–28.08.2015, Cambridge, UK,
    - Winter School in Abstract Analysis, Section: Set Theory and Topology, 25.01–1.02.2014, Hejnice, Czech Republic,
    - Workshop on Set Theory and its Applications to Topology and Real Analysis, in memory of Irek Recław, 4.07–6.07.2013, Gdańsk, Poland,
    - Winter School in Abstract Analysis, Section: Set Theory and Topology, 25.01–1.02.2013, Hejnice, Czech Republic,
    - Trends in set theory, 8.07–11.07.2012, Warsaw, Poland,
- 3 presentations on national scientific conferences: Real Analysis Workshops organized by Łódź University of Technology in Konopnica (in years 2015, 2016 and 2018),
- 1 presentation upon invitation at a foreign university: Košice set theory & topology seminar, Pavol Jozef Šafárik University in Košice, Slovakia (in year 2021),
- 8 presentations upon invitation at national universities:
    - 4 at Set Theory Seminar, University of Warsaw (in years 2015, 2016, 2020 and 2023),
    - 3 at Real Analysis Seminar, Łódź University of Technology (in years 2013, 2016 and 2021),
    - 1 at Set Theory Seminar, University of Wrocław (in year 2021).

I have also presented one poster at an international conference (New directions in the higher infinite, 10–14.07.2017, Edinburgh, UK).

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