SUMMARY OF PROFESSIONAL ACCOMPLISHMENT

1. Name

BARBARA WOLNIK

2. Diplomas, degrees conferred the degree, year of degree conferment, title of the PhD dissertation

Obtained diplomas and academic degrees:

- M.Sc.in Mathematics: The University of Wrocław, Faculty of Mathematics, Physics and Chemistry, 1990, title: *Representations of groups connected with certain Schrödinger operators*, supervisor: prof. Andrzej Hulanicki
- Ph.D. in Mathematics: The University of Gdańsk, Faculty of Mathematics and Physics, 2000, title: *About a certain class of bases in function spaces*, supervisor: prof. Anna Kamont

3. Information on employment in research institutes or faculties/departments or school of arts

- 1990 – 1991 – Trainee Assistant, The University of Wrocław, Faculty of Mathematics, Physics and Chemistry
- 1991 – 2000 – Assistant, The University of Gdańsk, Faculty of Mathematics and Physics (during the academic year 1992/1993 a maternity leave)
- 2000 – 2012 – Assistant Professor, The University of Gdańsk, Faculty of Mathematics and Physics (during the academic year 2001/2002 a maternity leave)
- 2012 – 2017 – Senior Lecturer, The University of Gdańsk, Faculty of Mathematics, Physics and Informatics
- since 2017 – Assistant Professor, The University of Gdańsk, Faculty of Mathematics, Physics and Informatics

4. Description of the achievements, set out in art. 219 para 1 point 2 of the Act

4.1 My main achievement

The main achievement set out in art. 219 para 1 point 2 of the Act is a monothematic series of publications under the title:

"Multidimensional number-conserving cellular automata with the von Neumann neighborhood"

List of publications included in the above-mentioned achievement:

[H1] Barbara Wolnik, Adam Dzedzej, Jan M. Baetens, Bernard De Baets
Number-conserving cellular automata with a von Neumann neighborhood of range one
Journal of Physics A-Mathematical and Theoretical, vol. 50 (43), 2017, 435101 (19 pages),
DOI:10.1088/1751-8121/aa89cf, 30 points, IF(1.963)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs in Section 4. The remaining results (Section 3) and the preparation of the manuscript are the outcome of joint work.

[H2] Barbara Wolnik, Anna Nenca, Jan M. Baetens, Bernard De Baets
A split-and-perturb decomposition of number-conserving cellular automata
Physica D-Nonlinear Phenomena, vol. 413, 2020, 132645 (12 pages),
DOI:10.1016/j.physd.2020.132645, 100 points, IF(1.807)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs. The applications presented in the article, in particular those requiring advanced computer simulations, as well as the preparation of the manuscript are the results of joint work.

[H3] Barbara Wolnik, Bernard De Baets
All binary number-conserving cellular automata based on adjacent cells are intrinsically one-dimensional
Physical Review E, vol. 100 (2), 2019, 022126 (6 pages),
DOI:10.1103/PhysRevE.100.022126, 140 points, IF(2.296)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs in Section 2 and Section 3. The preparation of the manuscript is the results of joint work.

[H4] Barbara Wolnik, Bernard De Baets
Ternary reversible number-conserving cellular automata are trivial
Information Sciences, Elsevier Inc., vol. 513, 2020, pp. 180-189,
DOI:10.1016/j.ins.2019.10.068, 200 points, IF(5.91)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs in Section 2 and Section 3. The preparation of the manuscript is the results of joint work.

[H5] Adam Dzedzej, Barbara Wolnik, Anna Nenca, Jan M. Baetens, Bernard De Baets
Efficient enumeration of three-state two-dimensional number-conserving cellular automata
Information and Computation, vol. 274, 2020, 104534 (14 pages),
DOI:10.1016/j.ic.2020.104534, 70 points, IF(0.872)

I am the author of Theorem 3.1 with the proof. The remaining results and the preparation of the manuscript are the outcome of joint work.

[H6] Adam Dzedzej, Barbara Wolnik, Maciej Dziemiańczuk, Anna Nenca, Jan M. Baetens, Bernard De Baets
A two-layer representation of four-state reversible number-conserving 2D cellular automata
Journal of Statistical Mechanics-Theory and Experiment, vol. 2019, 2019, 073202 (17 pages),
DOI:10.1088/1742-5468/ab25df, 70 points, IF(2.215)

This paper was created as a result of joint discussions. I am the author of the theoretical background in Section 4 (without algorithms). The preparation of the manuscript is the outcome of joint work.

[H7] Barbara Wolnik, Nikodem Mrożek, Adam Dzedzej, Bernard De Baets
Three-dimensional rotation-symmetric number-conserving cellular automata
Journal of Cellular Automata, vol. 15 (4), 2020, pp. 243-259, 40 points, IF(0.596)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs in Section 4. All theorems in Section 3 and Section 4 are the result of joint work (I estimate my contribution to be at least 50%). The preparation of the manuscript is the outcome of joint work.

[H8] Adam Dzedzej, Barbara Wolnik, Anna Nenca, Jan M. Baetens, Bernard De Baets
Two-dimensional rotation-symmetric number-conserving cellular automata
Information Sciences, Elsevier Inc., vol. 577, 2021, pp. 599-621,
DOI:10.1016/j.ins.2021.06.041, 200 points, IF(5.91)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs in Section 3 and in Appendix. The preparation of the manuscript is the outcome of joint work.

Outline of the research topic

Generally speaking, my research work concerns cellular automata (CAs), *i.e.* discrete dynamical systems that consist of a regular grid of cells that in consecutive time steps update their states depending on the states of their neighbors, according to some local rule.

When Ulam and von Neumann introduced cellular automata, they were motivated by biological applications and wanted to create a self-replicating machine, which could be analogous to the human brain and be computationally universal [1, 2]. Later on, CAs became of interest as models of complex phenomena in various fields of research, such as biology [3], environmental sciences [4], materials science [5], pedestrian dynamics [6], urban transport [7], hydrology [8] and agriculture [9], to name but a few. This popularity is due to the fact that CAs reflect the assumption that all laws (physical, economic, sociological and so on) must result from interactions that are strictly local.

More formally, a d-dimensional cellular automaton is a quadruple $(\mathcal{C}, Q, V, f)$, where

- $\mathcal{C} \subseteq \mathbb{Z}^d$ is the cellular space;
- Q is the finite set of states and we assume that Q is represented by numbers;
- $V = (\overrightarrow{\mathbf{v}_1}, \overrightarrow{\mathbf{v}_2}, \ldots, \overrightarrow{\mathbf{v}_m})$ is the neighborhood vector, which means that for any cell $\mathbf{i} \in \mathcal{C}$ the cells $\mathbf{i} + \overrightarrow{\mathbf{v}_1}, \mathbf{i} + \overrightarrow{\mathbf{v}_2}, \ldots, \mathbf{i} + \overrightarrow{\mathbf{v}_m}$ are its neighbors;
- $f : Q^m \to Q$ is the local rule: the new state of a cell $\mathbf{i}$ is given by $f(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are the current states of the cells $\mathbf{i} + \overrightarrow{\mathbf{v}_1}, \mathbf{i} + \overrightarrow{\mathbf{v}_2}, \ldots, \mathbf{i} + \overrightarrow{\mathbf{v}_m}$, respectively.

Usually, a cellular space equals the entire space $\mathbb{Z}^d$ or some rectangular grid of cells with periodic boundary conditions, *i.e*

$$\mathcal{C} = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \ldots \times (\mathbb{Z}/n_d\mathbb{Z})$$
$$= \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\} \times \ldots \times \{0, 1, \ldots, n_d - 1\}.$$

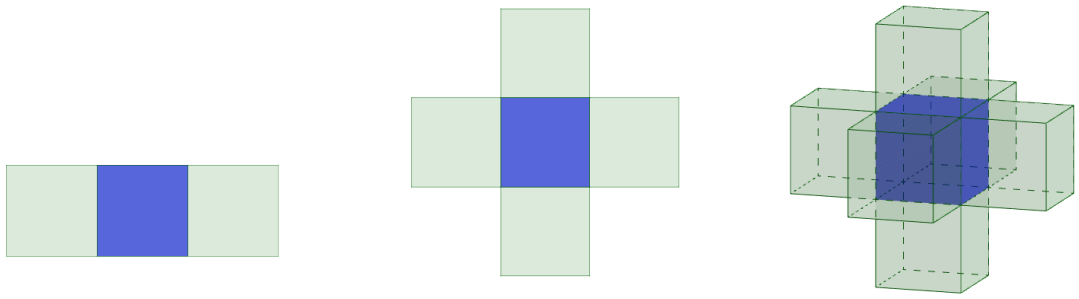
As for the neighborhood, there are two of the most commonly used types of the neighborhood: the Moore neighborhood and the von Neumann neighborhood. They differ in terms of which metric is considered in $\mathbb{Z}^d$: the Moore neighborhood is defined by the Chebyshev distance

$$\mathrm{dist}_{Ch}(\mathbf{i}, \mathbf{j}) = \max_{1 \leqslant k \leqslant d} |i_k - j_k|, \tag{1}$$

while the von Neumann neighborhood can be described using the Manhattan distance:

$$\mathrm{dist}_{M}(\mathbf{i}, \mathbf{j}) = \sum_{k=1}^{d} |i_k - j_k|, \tag{2}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathcal{C}$ and $\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathcal{C}$. From this point of view, these neighborhoods of a cell $\mathbf{i}$ can be defined as the closed unit balls centered at $\mathbf{i}$, *i.e.* the neighborhood consists of those cells whose distance (Chebyshev or Manhattan) from $\mathbf{i}$ is not greater than 1. While the Moore neighborhood has a very simple structure (each ball is a d-dimensional cube), the von Neumann neighborhood is a much more interesting and difficult object to study. For example, the von Neumann neighborhood in the case of a one-, two-, and three-dimensional grid is shown in Fig. 1. From now on we assume that we consider the von Neumann neighborhood only.



(a) (b) (c)

Figure 1: The von Neumann neighborhood in the case of (a) $d = 1$ (b) $d = 2$ (c) $d = 3$.

By a *configuration*, we mean any mapping from the grid $\mathcal{C}$ to Q and we denote the set of all configurations by $X = Q^{\mathcal{C}}$. If $|\mathcal{C}| < +\infty$, then for a given configuration $\mathbf{x} \in X$, its density is defined as:

$$\rho(\mathbf{x}) = \frac{1}{|\mathcal{C}|}\sigma(\mathbf{x}), \quad \text{where} \quad \sigma(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{C}} \mathbf{x}(\mathbf{i}).$$

By a *neighborhood configuration* we mean any function $N : V \to Q$. The set of all possible neighborhood configurations is denoted by $\mathcal{N}$. In the case of the von Neumann neighborhood, the set V has $2d + 1$ elements, namely $-\vec{\mathbf{v}}_d$, ..., $-\vec{\mathbf{v}}_1$, $\vec{\mathbf{0}}$, $\vec{\mathbf{v}}_1$, ..., $\vec{\mathbf{v}}_d$, where $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_d\}$ is the standard base. Thus, in this case, we can define any neighborhood configuration by the sequence

$$N(-\vec{\mathbf{v}}_d), \ldots, N(-\vec{\mathbf{v}}_1), N(\vec{\mathbf{0}}), N(\vec{\mathbf{v}}_1), \ldots, N(\vec{\mathbf{v}}_d)$$

(often without commas, unless confusion is possible). In the two-dimensional case, we prefer, however, to represent N graphically as $\begin{smallmatrix} & q_1 & \\ q_2 & q_3 & q_4 \\ & q_5 & \end{smallmatrix}$, with $N(\vec{\mathbf{0}}) = q_3$, $N(\vec{\mathbf{v}}_1) = q_4$, $N(-\vec{\mathbf{v}}_1) = q_2$, $N(\vec{\mathbf{v}}_2) = q_1$ and $N(-\vec{\mathbf{v}}_2) = q_5$. Using this notation, the set $\mathcal{N}$ can be expressed as:

$$\mathcal{N} = \left\{ \begin{smallmatrix} & q_1 & \\ q_2 & q_3 & q_4 \\ & q_5 & \end{smallmatrix} \mid q_1, q_2, q_3, q_4, q_5 \in Q \right\}.$$

Similarly, for $d = 3$, we write $\begin{smallmatrix} & q_1 & q_7 \\ q_2 & q_3 & q_4 \\ q_6 & q_5 & \end{smallmatrix}$, with $N(\vec{\mathbf{0}}) = q_3$, $N(\vec{\mathbf{v}}_1) = q_4$, $N(-\vec{\mathbf{v}}_1) = q_2$, $N(\vec{\mathbf{v}}_2) = q_1$, $N(-\vec{\mathbf{v}}_2) = q_5$, $N(\vec{\mathbf{v}}_3) = q_6$ and $N(-\vec{\mathbf{v}}_3) = q_7$.

If $\mathbf{x} \in X$ and $\mathbf{i} \in \mathcal{C}$ are given, then $N_{\mathbf{x},\mathbf{i}}$ denotes the configuration of the von Neumann neighborhood of cell $\mathbf{i}$ in the configuration $\mathbf{x}$, *i.e.* $N_{\mathbf{x},\mathbf{i}}(\vec{\mathbf{v}}) = \mathbf{x}(\mathbf{i} + \vec{\mathbf{v}})$, for each $\vec{\mathbf{v}} \in V$.

Any function $f : \mathcal{N} \to Q$ is called a *local rule*. Each local rule f induces a *global rule* $A_f : X \to X$ defined for $\mathbf{x} \in X$ and $\mathbf{i} \in \mathcal{C}$ as follows

$$A_f(\mathbf{x})(\mathbf{i}) = f(N_{\mathbf{x},\mathbf{i}}).$$

We identify the global rule A_f with the cellular automaton.

In my investigation, I focus on d-dimensional CAs that have two important (from the point of view of applications) properties. The first one is number conservation, which means that the sum of all states in any configuration remains constant throughout the evolution of the automaton. In the case of a finite $\mathcal{C}$ the following definition is used:

Definition 1 *A local rule f is* number-conserving *if its corresponding global rule A_f conserves the density. In other words, $\rho(A_f(\mathbf{x})) = \rho(\mathbf{x})$ holds for every configuration $\mathbf{x} \in X$. Equivalently, this condition can be expressed as $\sigma(A_f(\mathbf{x})) = \sigma(\mathbf{x})$.*

Note that if the state set contains not only natural numbers, then the term *density-conserving* is preferred.

The second property of CAs I am interested in is reversibility. Thanks to Hedlund's paper [10], this concept can be defined in several ways, but for my purposes the most useful is the following:

Definition 2 *Let f be the local rule of some d-dimensional CA. We say that this CA is reversible if the function A_f is an injection.*

In recent years, scientists are more and more apt to use multidimensional CAs. In particular, multidimensional CAs seem to be an appropriate tool to model physical phenomena, especially as a model of a system of interacting particles moving in a lattice [11–13] and, for example, in [14] a CA interpretation of quantum mechanics based on multidimensional CAs is presented.

Unfortunately, with the increase in dimension the set of all CAs grows rapidly. As a consequence, conventional methods, such as scanning through the entire set to find the CAs one is interested in, are no longer applicable. Hence, developing new tools for multidimensional CAs has become of utmost importance.

In view of incorporating conservation laws, a key requirement in physics, many models are based on a particular type of CAs, namely those that have the special feature of preserving a fixed invariant. Most often, these are the CAs that keep the sum of the states upon every update. Such CAs, called as already mentioned "number-conserving" and denoted NCCA (or "density-conserving" when non-integer states are allowed), have received ample attention, especially, as it is mentioned above, as models of systems of particles that interact and move along some predefined lattice: they appear in the natural context of gas or fluid flow [15,16] and highway traffic [17–20].

The von Neumann neighborhood is a natural choice when modelling physical phenomena (for example, it is "the most common in overland flow models" [8]). Unfortunately, studying multidimensional CAs with this kind of neighborhood is very complicated, because it is not a Cartesian product of one-dimensional neighborhoods (in contrast to the Moore neighborhood). For this reason, the problem of number conservation in d-dimensional CAs with the von Neumann neighborhood has been poorly investigated for $d > 1$.

Obviously, for a given CA, one does not need new tools to determine whether it is number-conserving or not. Boccara and Fukś [21] gave necessary and sufficient conditions for a one-dimensional CA to be an NCCA. Similarly, Durand et al. [22] gave such conditions in two or more dimensions. In the latter work the Moore neighborhood is considered, so the results can be used for the von Neumann neighborhood as well. However, if one wants to find all NCCAs for a given d and Q, it is, in general, impossible to check all CAs to find the number-conserving ones, due to the huge cardinality of the search space. In particular, it is not advisable to consider the von Neumann neighborhood as a subset of the Moore neighborhood as the first one has only $2d + 1$ cells, while the second one has as many as 3^d cells.

Since NCCAs often arise when modelling systems of indestructible particles, moving according to certain rules, several researchers have attempted to represent NCCAs in terms of particle displacements. In the one-dimensional case, this problem was tackled by Boccara and Fukś [21], Pivato [23] and Moreira et al. [12]. Kari and Taati [24] showed that also the dynamics of two-dimensional NCCAs can be described in terms of such particle displacements.

Tanimoto and Imai [25] gave a characterization of two-dimensional NCCAs with the von Neumann neighborhood in terms of so-called *flow functions* defined along three directions (vertical, horizontal and diagonal). This characterization allows to create two-dimensional NCCAs. Unfortunately, due to the computational complexity, it is still not of much use to find all two-dimensional NCCAs, even in the simple case of $Q = \{0, 1, 2\}$. However, using these flow functions Tanimoto and Imai succeeded in describing all two-dimensional five-state NCCAs with the von Neumann neighborhood that are rotation-symmetric, *i.e.*, are invariant under rotation of the

neighborhood by 90 degrees [26]. The results presented in [25,26] concern only $d = 2$, and the ideas used therein, in particular the flow functions, have not been transferred to higher dimensions. Even if we could use similar tools for $d > 2$, the results would be of no practical value, while using them to find all NCCAs would require computational power beyond current technical capabilities.

It is worth emphasizing that from the theoretical point of view, the case of $\mathcal{C} = \mathbb{Z}^d$ is the most interesting, but if we want to be able to simulate cellular automata on computers, we must limit ourselves to the case of finite grids (usually with periodic boundary conditions, but not only). Since for a given local rule properties of the cellular automaton considered on finite grids can be completely different than on the infinite grid, one should keep in mind that experiments done by computers may sometimes be misleading. This is why developing theoretical mathematical tools is so important (what I am doing in my research work).

Description of my main achievement

In the series of papers [H1 - H8], I study d-dimensional NCCAs with the von Neumann neighborhood, where d is a positive integer greater than 1. Papers [H1 - H4] concern the theory for a general d, while papers [H5 - H8] contain results of some particular investigation for $d = 2$ or $d = 3$. I present a detailed description of each of these papers below.

[H1]

The first results of my efforts to find a convenient characterization of multidimensional number-conserving cellular automata with the von Neumann neighborhood are presented in the paper *"Number-conserving cellular automata with a von Neumann neighborhood of range one"* [27]. Using a novel approach based on a geometric analysis of the von Neumann neighborhood in higher dimensions, I managed to find necessary and sufficient conditions for a d-dimensional CA to be number conserving, formulated in terms of the local rule in a way similar to the one presented by Boccara and Fukś in [21]. Moreover, these conditions apply for any state set $Q \subset \mathbb{R}$, whether it is finite or not.

The main idea was to reduce the notation of the condition to the simplest possible neighborhood configurations: *monomers* M and *dimers* D, *i.e.* functions $M_{\overrightarrow{\mathbf{v}}:q} \colon V \to Q$ and $D_{\substack{\overrightarrow{\mathbf{u}}:p \\ \overrightarrow{\mathbf{w}}:q}} \colon V \to Q$ given by the following formulas:

$$M_{\overrightarrow{\mathbf{v}}:q}(\overrightarrow{\mathbf{u}}) = \begin{cases} q & \text{, if } \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} \\ 0 & \text{, if } \overrightarrow{\mathbf{u}} \neq \overrightarrow{\mathbf{v}} \end{cases} \qquad D_{\substack{\overrightarrow{\mathbf{u}}:p \\ \overrightarrow{\mathbf{w}}:q}}(\overrightarrow{\mathbf{v}}) = \begin{cases} p & \text{, if } \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{u}} \\ q & \text{, if } \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{w}} \\ 0 & \text{, otherwise} \end{cases} .$$

The main result of [H1] states that CA with the von Neumann neighborhood preserves the sum of states if and only if for each neighborhood configuration N it holds that

$$\begin{aligned} f(N) = N(\overrightarrow{\eta}) &+ \sum_{(\overrightarrow{\mathbf{u}},\overrightarrow{\mathbf{w}})\in\mathbf{\Lambda}} \left[f\left(D_{\substack{\overrightarrow{\mathbf{u}}:N(\overrightarrow{\mathbf{u}}) \\ \overrightarrow{\mathbf{w}}:N(\overrightarrow{\mathbf{w}})}} \right) - f\left(D_{\substack{\overrightarrow{\mathbf{u}}:N(\text{-}\overrightarrow{\mathbf{w}}) \\ \overrightarrow{\mathbf{w}}:N(\text{-}\overrightarrow{\mathbf{u}})}} \right) \right] \\ &- \sum_{(\overrightarrow{\mathbf{u}},\overrightarrow{\mathbf{w}})\in\mathbf{\Lambda}} \left[f^E\left(D_{\substack{\overrightarrow{\mathbf{u}}:N(\overrightarrow{\mathbf{u}}) \\ \overrightarrow{\mathbf{w}}:N(\overrightarrow{\mathbf{w}})}} \right) + f^E\left(D_{\substack{\text{-}\overrightarrow{\mathbf{w}}:N(\overrightarrow{\mathbf{u}}) \\ \text{-}\overrightarrow{\mathbf{u}}:N(\overrightarrow{\mathbf{w}})}} \right) \right] \\ &+ \sum_{\overrightarrow{\mathbf{v}}\in V\setminus\{\overrightarrow{\eta}\}} f^E\left(H_{N(\overrightarrow{\mathbf{v}})} \right) - \sum_{\overrightarrow{\mathbf{v}}\in V_+} f\left(M_{\overrightarrow{\mathbf{v}}:N(\text{-}\overrightarrow{\mathbf{v}})} \right) , \end{aligned} \tag{3}$$

where

- $\mathbf{\Lambda} \subset V \times V$ is a certain finite set of selected pairs of directions (very easy to describe);
- the *monomer expansion* $f^E \colon \mathcal{N} \to \mathbb{R}$ of a local rule f is defined by

$$f^E(N) = \sum_{\overrightarrow{\mathbf{v}}\in V} f\left(M_{\overrightarrow{\mathbf{v}}:N(\overrightarrow{\mathbf{v}})} \right) ;$$

- and for a given $q \in Q$, H_q denotes the homogeneous neighborhood configuration, *i.e.* for all $\overrightarrow{\mathbf{v}} \in V$ it holds that $H_q(\overrightarrow{\mathbf{v}}) = q$.

Although this formula is rather lengthy, it is much simpler than conditions presented by other authors (cf. [22] and [25]). The greatest advantage of this formula is that its form not only allows to decide whether a given rule is number-conserving, but also to enumerate all number-conserving rules for not too large d and small $|Q|$. This is due to the fact that we do not have to consider every rule from a huge space of $|Q|^{|Q|^{2d+1}}$ possible rules one by one, but we only need to set at most $(2d+1)\cdot(|Q|-1)$ monomers and $d^2(|Q|-1)^2$ dimers and then verify whether the outputs calculated according to Eq. (3) belong to Q. For example, we were able to use it to find out that if $d=3$, then there are only 13 number-conserving binary rules, namely: the identity rule, 6 shift rules and 6 traffic rules (in each of the possible directions: right and left, up and down, forward and backward). Note that all these rules are in some sense trivial, since they can be considered as one-dimensional rules acting on disjoint strips of cells (a generalization of this result is contained in [H3]). The main result presented in [H1] allowed also to find all two-dimensional three-state NCCAs (see [H6]) and to describe all affine continuous density-conserving CAs with the infinite state set $Q=[0,1]$ (see [28]).

Additionally, formula (3) has another important advantage: for a given d there are exactly $(2d+1)\cdot 2^{d^2}$ possible formulations of this necessary and sufficient condition. This is because there are $2d+1$ possibilities for choosing the leading term $\overrightarrow{\eta}$ and 2^{d^2} possibilities for choosing $\mathbf{\Lambda}$. Although they all are equivalent, the obtained formulas can differ in the number of terms. For example, for $d=2$ formula (3) can be written in 80 different ways and the shortest one is the following:

$$\begin{aligned} f\begin{pmatrix} & q_1 & \\ q_2 & q_3 & q_4 \\ & q_5 & \end{pmatrix} = q_1 &+ f\begin{pmatrix} & 0 & \\ q_2 & 0 & 0 \\ & q_5 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ q_1 & 0 & 0 \\ & q_4 & \end{pmatrix} + f\begin{pmatrix} & 0 & \\ 0 & 0 & q_4 \\ & q_5 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & 0 & q_1 \\ & q_2 & \end{pmatrix} \\ &+ f\begin{pmatrix} & 0 & \\ 0 & q_3 & q_4 \\ & 0 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & q_2 & q_3 \\ & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & \\ 0 & q_3 & 0 \\ & q_5 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & q_1 & 0 \\ & q_3 & \end{pmatrix} \\ &+ f\begin{pmatrix} & 0 & \\ 0 & q_2 & 0 \\ & 0 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & q_3 & 0 \\ & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & \\ 0 & 0 & q_3 \\ & 0 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & 0 & q_4 \\ & 0 & \end{pmatrix} \\ &+ f\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & q_2 & \end{pmatrix} + f\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & q_3 & \end{pmatrix} + f\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & q_4 & \end{pmatrix} - f\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & q_1 & \end{pmatrix} - 2f\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & q_5 & \end{pmatrix}. \end{aligned} \tag{4}$$

Formulas similar to (4), but for the case $d=3$ are a bit more complicated and the right-hand side of the shortest expression has 38 elements in the sum.

Theorem 3 *Consider dimension $d=3$. A local rule f is number-conserving if and only if for any $q_1,q_2,q_3,q_4,q_5,q_6,q_7\in Q$, it holds that*

$$\begin{aligned} f\begin{pmatrix} q_1 & q_7 & \\ q_2 & q_3 & q_4 \\ q_6 & q_5 & \end{pmatrix} = q_1 &+ f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & q_4 \\ 0 & q_5 & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & q_1 \\ 0 & q_2 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & q_3 & 0 \\ 0 & q_5 & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & \\ 0 & q_1 & 0 \\ 0 & q_3 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ q_2 & 0 & 0 \\ 0 & q_5 & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & \\ q_1 & 0 & 0 \\ 0 & q_4 & \end{pmatrix} \\ &+ f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ q_6 & q_5 & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ q_1 & q_7 & \end{pmatrix} + f\begin{pmatrix} 0 & q_7 & \\ 0 & 0 & 0 \\ 0 & q_5 & \end{pmatrix} - f\begin{pmatrix} 0 & q_1 & \\ 0 & 0 & 0 \\ 0 & q_6 & \end{pmatrix} + f\begin{pmatrix} 0 & q_7 & \\ 0 & 0 & q_4 \\ 0 & 0 & \end{pmatrix} - f\begin{pmatrix} 0 & q_2 & \\ 0 & 0 & q_6 \\ 0 & 0 & \end{pmatrix} \\ &+ f\begin{pmatrix} 0 & q_7 & \\ 0 & q_3 & 0 \\ 0 & 0 & \end{pmatrix} - f\begin{pmatrix} 0 & q_3 & \\ 0 & q_6 & 0 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_4 \\ q_6 & 0 & \end{pmatrix} - f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_7 \\ q_2 & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & 0 \\ 0 & q_3 & q_4 \\ 0 & 0 & \end{pmatrix} - f\begin{pmatrix} & 0 & 0 \\ 0 & q_2 & q_3 \\ 0 & 0 & \end{pmatrix} \\ &+ f\begin{pmatrix} 0 & 0 & \\ 0 & q_2 & 0 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & q_6 & 0 \\ 0 & 0 & \end{pmatrix} - 2f\begin{pmatrix} 0 & 0 & \\ 0 & q_3 & 0 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_3 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_6 \\ 0 & 0 & \end{pmatrix} \\ &+ f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_7 \\ 0 & 0 & \end{pmatrix} - 3f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & q_4 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_2 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_3 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_4 & \end{pmatrix} \\ &+ f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_6 & \end{pmatrix} + f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_7 & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_1 & \end{pmatrix} - 4f\begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & q_5 & \end{pmatrix} + f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ q_2 & 0 & \end{pmatrix} \\ &- f\begin{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ q_6 & 0 & \end{pmatrix} + f\begin{pmatrix} 0 & q_2 & \\ 0 & 0 & 0 \\ 0 & 0 & \end{pmatrix} + f\begin{pmatrix} 0 & q_3 & \\ 0 & 0 & 0 \\ 0 & 0 & \end{pmatrix} - 2f\begin{pmatrix} 0 & q_7 & \\ 0 & 0 & 0 \\ 0 & 0 & \end{pmatrix}. \end{aligned}$$

And again, the formula in the above theorem is only one of as many as $7\cdot 2^9$ possible formulations of this necessary and sufficient condition.

Thus in each particular case we can choose the version that suits us best. This method is especially useful to describe number-conserving rules satisfying some additional conditions – in particular in cases where these additional conditions result in dependencies between monomers or dimers. This happens, for example, in the

case of local rules with some kind of symmetry, the most natural one being rotation symmetry (it will be the topic of [H7] and [H8]).

Personally, I rate the result in [H1] more artisanal than artistic. In fact, it is obtained thanks to the great determination in trying to translate the known one-dimensional result into more dimensions and to overcome the associated technical problems. I am not saying that it is trivial (as much as 15 years have passed since the one-dimensional case was proved by Boccara and Fukś), but there is no revolutionary concept here.

[H2]

As I mentioned above, the necessary and sufficient conditions presented in [29] ([H1]) can be formulated in $(2d+1)2^{d^2}$ different forms, where d is the considered dimension, and although they all are equivalent, the obtained formulas can differ in the number of terms. A better understanding of this fact has led me to the idea of a completely new approach to the study of number conservation. In *"A split-and-perturb decomposition of number-conserving cellular automata"* [30] I proved that the local rule of any NCCA with the von Neumann neighborhood can be decomposed into two parts: a split function and a perturbation. Moreover, the set of all possible split functions has a very simple structure, while the set of all perturbations forms a linear space and is therefore very easy to describe in terms of a basis.

To develop the idea of decomposition, it is important to consider *local rules in a wider sense*: we abandon the assumption that the values of a local rule must belong to Q. Such objects we call *local functions*.

Definition 4 *A local function h is called a* **split function** *if it satisfies:*

(S1) $h(M_{\overrightarrow{\mathbf{v}}:q}) \in Q$, *for any monomer* $M_{\overrightarrow{\mathbf{v}}:q}$;

(S2) $\sum_{\overrightarrow{\mathbf{v}} \in V} h(M_{\overrightarrow{\mathbf{v}}:q}) = q$, *for any state* $q \in Q$;

(S3) $h(N) = \sum_{\overrightarrow{\mathbf{v}} \in V} h(M_{\overrightarrow{\mathbf{v}}:N(\overrightarrow{\mathbf{v}})})$, *for any* $N \in \mathcal{N}$.

The set of all split functions is denoted by $\mathcal{S}$.

The above definition reflects an obvious fact that if an initial configuration has only one cell with a nonzero state, then in the next time step a number-conserving CA must redistribute this state to the cells located in the neighborhood in such a way that the redistributed parts belong to the state set Q. In other words, this state splits, which happens according to some recipe depending on the state. Split functions act as follows: each state splits according to its recipe irrespective of the states of its neighbors. Of course, split functions do not have to be local rules. Indeed, as each state splits independently, it may happen that the sum of the splitted constituents ending up in one cell from different neighbors does not belong to the state set Q. However, this is a small disadvantage compared to the benefits they bring along.

Note that the set $\mathcal{S}$ has a very simple structure. Indeed, it can be identified with the Cartesian product $\bigtimes_{q \in Q_+} S_q$, where $Q_+ = Q \backslash \{0\}$ and

$$S_q = \left\{(x_1, x_2, \ldots, x_{2d+1}) \mid x_1 + x_2 + \ldots + x_{2d+1} = q,\ x_1, x_2, \ldots, x_{2d+1} \in Q\right\}.$$

In particular, if Q is finite, the cardinality of the set $\mathcal{S}$ equals $\prod_{q \in Q_+} |S_q|$. For example, in the case of the so-called k-ary CAs (binary, ternary, quaternary, quinary, and so on), in which the state set equals $\{0, 1, \ldots, k-1\}$, for some integer $k \geqslant 2$ (see, for instance, [31–34]), for each $q \in \{1, \ldots, k-1\}$ it holds that $|S_q| = \binom{2d+q}{q}$ (see e.g. [35]), so, we have

$$|\mathcal{S}| = \binom{2d+1}{1}\binom{2d+2}{2} \cdot \ldots \cdot \binom{2d+k-1}{k-1}. \tag{5}$$

Table 1 presents $|\mathcal{S}|$ for certain cases of dimension d and state sets Q, while in the brackets the number of all local rules is given. One can see that the total number of split functions is definitely less than the number of all local rules. For example, in the two-dimensional case for $Q = \{0, 1, 2\}$, there are only 75 split functions, while there are 3^{3^5} local rules – the last number has 116 digits.

It appears that for a number-conserving local rule the only possible derogation from being a split function is so-called *perturbation*.

Definition 5 *A local function $g : \mathcal{N} \to \mathbb{R}$ is called a* **perturbation** *if it satisfies the following two conditions:*

(P1) *for any $\overrightarrow{\mathbf{v}} \in V$ and for any $q \in Q$, it holds that $g(M_{\overrightarrow{\mathbf{v}}:q}) = 0$,*

	$d=1$	$d=2$	$d=3$	$d=4$
$Q=\{0,1\}$	3 (256)	5 (2^{32})	7 (2^{128})	9 (2^{512})
$Q=\{0,1,2\}$	18 (3^{27})	75 (3^{243})	196 (3^{2187})	405 (3^{19683})
$Q=\{0,1,2,3\}$	180 (4^{64})	2625 (4^{1024})	16464 (4^{16384})	66825 (4^{262144})

Table 1: The number of all split functions versus the number of all local rules for state set Q and dimension d.

(P2) *for any* $\mathbf{x} \in X$*, it holds that* $\sigma(A_g(\mathbf{x})) = 0$.

The set of all perturbations is denoted by $\mathcal{P}$.

The set of all perturbations also has a very nice structure – it is a linear space – and therefore it can be very easily described in terms of the elements of a basis of this space.

Theorem 6 *Let* $\mathbf{\Lambda}$ *be given. A local function* g *is a perturbation if and only if it satisfies* (P1) *of Definition 5 and*

$$g(N) = \sum_{\{\overrightarrow{\mathbf{u}},\overrightarrow{\mathbf{w}}\}\in\mathbf{\Lambda}} \left[g\left(D_{\substack{\overrightarrow{\mathbf{u}}:N(\overrightarrow{\mathbf{u}}) \\ \overrightarrow{\mathbf{w}}:N(\overrightarrow{\mathbf{w}})}} \right) - g\left(D_{\substack{\overrightarrow{\mathbf{u}}:N(-\overrightarrow{\mathbf{w}}) \\ \overrightarrow{\mathbf{w}}:N(-\overrightarrow{\mathbf{u}})}} \right) \right], \tag{6}$$

holds for any $N \in \mathcal{N}$.

As a consequence of Theorem 6 we obtain the following remark.

Remark 7 *To define a perturbation, it suffices to declare its value on dimers* $D_{\substack{\overrightarrow{\mathbf{u}}:p \\ \overrightarrow{\mathbf{w}}:q}}$*, where* $\{\overrightarrow{\mathbf{u}},\overrightarrow{\mathbf{w}}\} \in \mathbf{\Lambda}$ *and* $p, q \in Q_+$.

The main theorem of [H2] states the following:

Theorem 8 *A local rule* f *is number-conserving if and only if there exist a split function* h *and a perturbation* g *such that* $f = h + g$*. Moreover, for a given local rule* f*, the functions* h *and* g *are uniquely determined.*

The above result is the one I am most proud of. It is a solution to the problem of finding such characterization of number-conserving local rules, that is, firstly: based on a simple theory, and secondly: it will significantly facilitate or even enable the search for complete lists of number-conserving local rules for given d and Q.

From the mathematical point of view, it is a compilation of two very well known ideas. The first idea can be found, for example, in the topic of cubic equations. Indeed, if you need to solve a cubic equation over the field of real numbers you have no formulas to do this. But you can broaden the area of your search and find the roots over the field of complex numbers using Cardano formula (in the theory above split functions play a role of complex roots) and then choose the ones that are real (we have to modify split functions to local rules using perturbations). The second idea is the main idea in the theory of solving linear equations (for example, differential ones): to solve an inhomogeneous equation you need to find at least one of the solutions and then to find all solutions of the corresponding homogeneous equation. In the theory above, split functions are certain solutions of the inhomogeneous equation, while perturbations are all solutions of the corresponding homogeneous equation. Of course, these ideas can be compiled in various ways, but the goal was to define split functions and perturbations such that both sets $\mathcal{S}$ and $\mathcal{P}$ would have a simple structure, well suited for computer simulations.

The paper [H2] also includes examples showing how the decomposition theorem allows to reduce the computational complexity of finding all NCCAs with the von Neumann neighborhood for given d and Q. In particular, it was possible to compile a complete list of three-dimensional NCCAs with three states. Before introduction of the presented decomposition, this case had been beyond the capabilities of presently available computers (for example, using the theory from Durand et al. [22] would require checking $3^{3^7} \approx 2.9 \cdot 10^{1043}$ local rules). Moreover, this new approach to study d-dimensional NCCAs with the von Neumann neighborhood gave hope for solutions to some unresolved questions in the field, especially the following ones:

- Is it true that for any d there are no non-trivial binary NCCAs?
- Is it true that three states are too few to enable the existence of nontrivial reversible NCCAs with the von Neumann neighborhood in any dimension?

And indeed, the decomposition theorem allowed to find answers to both these questions (see [H3] and [H4]).

[H3]

In the paper *"All binary number-conserving cellular automata based on adjacent cells are intrinsically one-dimensional"* [36] I focus on the binary d-dimensional CAs with the von Neumann neighborhood. The physical interpretation of the considered CAs is very intuitive: we want to describe the movement of particles in a d-dimensional grid. Each cell of the grid is either empty or contains a particle. In subsequent time steps the particles move between the cells (in each time step a particle can move to an adjacent cell only), but in one cell there can be at most one particle at a time.

Before [H3] the vast majority of results on binary CAs concerned one-dimensional ones. This is connected with the double exponential increase of the number of possible local rules: if the dimension of the space is d, then the number of cells forming the von Neumann neighborhood equals $2d+1$, thus there are $2^{2^{2d+1}}$ binary CAs in this case. One-dimensional binary CAs, better known as Elementary CAs (ECAs), are fairly well researched and, in particular, it is known that among all 256 of them, there are only five that are number-conserving: the identity rule (ECA 204), the shift-left rule (ECA 170), the shift-right rule (ECA 240), the traffic-right rule (ECA 184) and the traffic-left rule (ECA 226) [37]. Increasing the dimension by one yields as many as 4 294 967 296 two-dimensional binary CAs, yet only four new NCCAs appear: the shift and the traffic rules in the downward and upward direction. If $d=3$, then the cardinality of the set of all CAs increases to the incredibly large number $2^{128} \approx 3.4 \times 10^{38}$, but using the characterization of number-conserving CAs given in [H1], it was still possible to find all NCCAs and we obtained that there are only 13, namely: the identity rule, six shift rules and six traffic rules (in each of the possible directions: right, left, up, down, forward and backward). In [30], the hypothesis was made that in the binary case, increasing the space dimension does not result in the appearance of a new kind of number-conserving CAs. The main result of [H3] confirm this hypothesis:

Theorem 9 *Let the dimension $d \geqslant 1$ be given. There are exactly $4\,d+1$ d-dimensional binary NCCAs with the von Neumann neighborhood: the identity rule, the shift rules and traffic rules in each of the $2\,d$ directions.*

In this way, the structure of the d-dimensional number-conserving binary CAs with the von Neumann neighborhood became fully unveiled and it appeared that regardless of the dimension d, all of these cellular automata are trivial, as they are intrinsically one-dimensional. Moreover, it became possible to describe in detail the dynamics of such automata on finite grids. If we are dealing with a shift rule (for example, in direction $\overrightarrow{\mathbf{v}_1}$), then the initial pattern of a configuration does not change, it only moves in the direction $\overrightarrow{\mathbf{v}_1}$ by one cell in each time step. So, although Theorem 9 states that such CA acts independently on every one-dimensional $\overrightarrow{\mathbf{v}_1}$-row of the grid $\mathcal{C}$ (*i.e.* a row extending in the direction $\overrightarrow{\mathbf{v}_1}$), in the case of a shift rule we observe a synchronous action as an aggregate result. The case of traffic rules is different. For example, the traffic rule in direction $\overrightarrow{\mathbf{v}_1}$ acts as follows. The pattern of each $\overrightarrow{\mathbf{v}_1}$-row quickly stabilizes. Subsequently, if in a given $\overrightarrow{\mathbf{v}_1}$-row the number of 1s is smaller than the number of 0s, then in every consecutive time step, the 1s seem to go one step in the direction $\overrightarrow{\mathbf{v}_1}$. If the number of 1s is greater than the number of 0s, then in every consecutive time step, the 0s seem to go one step in the direction -$\overrightarrow{\mathbf{v}_1}$. If the number of 1s equals the number of 0s, then the pattern stabilizes as an alternating sequence of 0s and 1s, which moves with time in the direction $\overrightarrow{\mathbf{v}_1}$.

I am really pleased with this result. First of all, Theorem 9 showed that the decomposition theorem can also be a convenient tool for proving some general properties concerning d-dimensional NCCAs (never seen before in the literature). Moreover, it is obtained through the use of basic mathematical tools, similarly as, for instance, the famous result of Land and Belew [38], which fully resolved the problem of existence of a perfect solution of the density classification problem.

[H4]

In the paper *"Ternary reversible number-conserving cellular automata are trivial"* [39] I focus on the ternary d-dimensional CAs with the von Neumann neighborhood (*i.e.* d-dimensional CAs with the state set $\{0,1,2\}$). I wanted to answer the question which out of the huge number of $3^{3^{2d+1}}$ such CAs are both number conserving and reversible, since nothing was known about their reversibility in the case $d>1$.

The reversibility of CAs has received significant attention practically from the beginning. This is due to the fact that many researchers have further developed the theory of CAs to use them as a suitable computational tool for the simulation of physical systems. Thus, as reversibility is one of the fundamental laws of physics at the microscopic scale, especially in quantum mechanics, CAs should preferabley exhibit the same property.

The first studies on the reversibility of CAs are due to Hedlund [10] and Richardson [40]. In the literature one can now find a number of results on this topic. Among them are results regarding infinite CAs as well as finite CAs with various boundary conditions. However, the studies on the reversibility of CAs concentrate only on one-dimensional ones and linear CAs with $\mathbb{Z}_m$ as state set.

When it comes to one-dimensional CAs, it is safe to say that the problem has been completely resolved. Indeed, an effective way to determine reversibility of infinite one-dimensional CAs was shown by Amoroso and Patt [41] and also by Di Gregorio and Trautteur [42] and Sutner [43]. For finite CAs, an algorithm to decide the reversibility was developed by Bhattacharjee and Das [33]. Moreover, there are many results concerning specific Elementary CAs (ECAs) (see, for example, [44–47]). However, it is worth emphasizing that existing tools do not

allow to enumerate all reversible one-dimensional CAs even in the case of few states (of course, except in the case of ECAs), because it is impossible to check all CAs with a given state set, due to the inhibitive cardinality of the search space.

While one-dimensional reversible CAs have been studied in detail and from many different points of view, the same cannot be said about two- or higher-dimensional ones. This is a consequence of the negative result of Kari [48] stating that there is no algorithm that can decide whether or not an arbitrary two-dimensional CA is reversible.

Another class of CAs (one- or higher-dimensional) for which researchers have found methods to determine reversibility, is the class of linear CAs (LCAs). The reason that LCAs have been extensively studied, especially with $\mathbb{Z}_m$ as state set, is that they are particularly amenable to theoretical analysis. Firstly, the linearity of local rules opens the door to the realm of linear algebra and one can use powerful tools like transition matrices, since an LCA is reversible if and only if its transition matrix is reversible (see, for example, [49–52]). Secondly, the fact that the state set is a finite commutative ring allows to use formal power series representations to state necessary and sufficient conditions for the reversibility of an LCA in terms of the coefficients of its local rule. This method has been initiated by Itô et al. [53] and intensively developed by other researchers (see, for example, [54–56]). Thus, also in the case of LCAs with the state set $\mathbb{Z}_m$, we can say that the problem of reversibility has been solved completely. In particular, LCAs have been thoroughly examined in terms of ergodic theory: it was shown that a reversible LCA is either a Bernoulli automorphism or non-ergodic [57]. More information about the reversibility problem of these types of CAs can be found in [58–60].

However, the transition matrix tool has fundamental limitations, since the matrix size depends on the number of cells, so, it is useless when we deal with infinite CAs and for finite CAs, every grid size has to be considered separately. Moreover, applying matrix theory for a ring $\mathbb{Z}_m$, which does not have to be a field, one has to be very careful (see [61] or [62]). Also, the linearity assumption as well as the use of a finite ring as state set are often not acceptable. For example, in particle motion modeling, one would like two plus one to be three instead of zero as in the case of the field $\mathbb{Z}_3$.

In [H4] using the decomposition theorem allowed to characterize all reversible ternary d-dimensional NCCAs with the von Neumann neighborhood. Note that in this case the state set equals $\{0, 1, 2\}$ without any "nice mathematical structure". The main result of [H4] states:

Theorem 10 *Let $d \geqslant 1$. Any reversible d-dimensional ternary NCCA is a shift (in some direction $\vec{\mathbf{v}} \in V$).*

Thus it turned out that all reversible d-dimensional ternary NCCAs are trivial: they are intrinsically 1-dimensional. It confirmed the hypothesis that three states are too few to enable the existence of nontrivial reversible NCCAs with the von Neumann neighborhood in any dimension.

[H5]

The paper *"Efficient enumeration of three-state two-dimensional number-conserving cellular automata"* [63] presents an approach for finding all two-dimensional NCCA with the von Neumann neighborhood based on the results proven in [H1]. (Note that this paper was written before the decomposition theorem was obtained and it concerns only $d = 2$.) The efficiency of this approach is demonstrated by enumerating and describing four special cases: totalistic, outer-totalistic, binary and ternary two-dimensional NCCAs with the von Neumann neighborhood. Although the results for the first three cases were rather known (but the last result was not), the approach introduced in [H1] allowed to obtain these results with virtually no effort.

The first result concerns totalistic two-dimensional NCCAs with the von Neumann neighborhood (a totalistic CA is a CA whose local rule depends only on the sum of the states of all cells in the neighborhood). It states that the only candidate for the local rule of such CA is the arithmetic mean of the five states in the neighborhood. This leads to the following conclusions:

Corollary 11 *There exists no totalistic two-dimensional k-ary NCCA with the von Neumann neighborhood, while there exists exactly one totalistic two-dimensional NCCA with state set $Q = [0, 1]$ and with the von Neumann neighborhood.*

The case of outer-totalistic two-dimensional NCCAs looks, unfortunately, not much better (an outer-totalistic CA is a CA whose local rule depends only on the state of the central cell and the sum of the states of the adjacent cells):

Theorem 12 *Let $k > 1$ be an integer. There exists only one outer-totalistic two-dimensional k-ary NCCA with the von Neumann neighborhood, namely the identity.*

Theorem 13 *Let f be the local rule of a two-dimensional CA with the von Neumann neighborhood with state set $Q = [0, 1]$. It is the local rule of an outer-totalistic two-dimensional NCCA if and only if there exists $\alpha \in [0, 1]$ such that*

$$f\begin{pmatrix} & q_1 & \\ q_2 & q_3 & q_4 \\ & q_5 & \end{pmatrix} = \alpha q_3 + \frac{1}{4}(1-\alpha)(q_1 + q_2 + q_4 + q_5)\,, \tag{7}$$

for any $q_1, q_2, q_3, q_4, q_5 \in [0, 1]$.

The second part of [H5] is devoted to two-dimensional ternary NCCAs with the von Neumann neighborhood. First of all, let us note that using Eq. (4) one can immediately find all two-dimensional binary NCCAs with the von Neumann neighborhood (we arrive at a system of 32 linear equations in 8 variables). It turns out that it has only 9 solutions in the set $\{0, 1\}$, so there are only 9 different two-dimensional binary NCCAs, namely: the identity rule, 4 shift rules and 4 traffic rules (in each of the possible directions: right, left, up and down). All of them are in some sense trivial, since they can be considered as one-dimensional rules acting either row- or columnwise.

A slightly more subtle use of Eq. (4) for $Q = \{0, 1, 2\}$ allows to find all 1327 two-dimensional ternary NCCAs and only 287 of them are one-dimensional CAs acting in either a vertical or horizontal direction. The set of all two-dimensional ternary NCCAs can be divided into 145 equivalence classes. Indeed, we can apply one of the 8 symmetries of the von Neumann neighborhood (along a vertical, horizontal or diagonal axis and rotation by 90, 180 or 270 degrees) and possibly the conjugation operator (exchanging states 0 and 2):

$$f^C\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = 2 - f\begin{pmatrix} & 2-u & \\ 2-l & 2-c & 2-r \\ & 2-d & \end{pmatrix}.$$

Thus there are only 145 independent two-dimensional ternary NCCAs. Among these CAs there are exactly 48 one-dimensional CAs described earlier in [21]. The remaining 97 CAs operate in a truly two-dimensional way. Thus, the case of ternary CAs is much more complex than that of binary CAs.

The last result of [H5] concerns the reversibility of two-dimensional ternary NCCAs. Deciding reversibility of a CA in two or higher dimensions is considered to be a hard problem (as I mentioned before, it was shown to be an undecidable property in general [48]), so I think that this result is the main result of [H5].

Theorem 14 *There are no nontrivial reversible two-dimensional ternary NCCAs with the von Neumann neighborhood.*

The above theorem shows that when it comes to the issue of reversibility of ternary NCCAs with the von Neumann neighborhood, then the case $d = 2$ does not differ from the case $d = 1$. While working on this theorem, it was hypothesized that increasing the dimension further does not allow any new type of reversible ternary NCCAs with the von Neumann neighborhood. As I already mentioned, it was then proven in [H2].

[H6]

The paper *"A two-layer representation of four-state reversible number-conserving 2D cellular automata"* [64] presents a method of enumerating all reversible two-dimensional quaternary NCCAs with the von Neumann neighborhood. It appears that each such CA can be envisaged as two mutually connected one-dimensional binary NCCAs. Developing a method to obtain this result was a major step in the study of multi-dimensional NCCAs.

Previously, research focused on one-dimensional case. Schranko and De Oliveira [65] performed numerical experiments involving many rules to finally conjecture that the class of reversible one-dimensional NCCAs is too restrictive to be computationally universal. This was proved in the case of radius 1/2 CAs [66], but it does not hold in general as shown by Morita [67]. Imai *et al.* [68] listed all reversible one-dimensional NCCAs with radius 1 and up to four states through an exhaustive search and they also constructed some five-state reversible one-dimensional NCCAs (but the list was not complete).

When it comes to two dimensions, it was proven in [63] ([H5]) that in the case of the state set $\{0, 1\}$ or $\{0, 1, 2\}$, *i.e.*, when we deal with binary or ternary reversible NCCAs, there are only trivial ones, namely, the identity and the shifts (in each of the four possible directions). However, there was no practically feasible approach to enumerate all reversible NCCAs with more than three states in two (or more) dimensions and finding such approach at least for two dimensions has been considered an intricate task because of two main reasons. Firstly, in general, reversibility of multi-dimensional CAs is undecidable [48]. Restrictions to finite grids are obviously decidable, but still computationally hard (see, e.g., [69]). Secondly, there were no tools available to enumerate all quaternary two-dimensional NCCAs - in the case of four states it becomes impractical to search the entire space of two-dimensional CAs using the necessary and sufficient condition for number conservation by Durand *et al.* [22] due to the size of the search space.

The method described in [H6] can overcome these two problems. Firstly, it starts with an innovative description of reversible one-dimensional quaternary NCCAs that envisages them as being constituted of two mutually connected layers, further on referred to as the two-layer representation. This two-layer view can easily be carried over to two dimensions and allows us to construct a list of as many as 65 reversible two-dimensional quaternary NCCAs with the von Neumann neighborhood (including 60 non-trivial ones), which are provably reversible. Secondly, we use the novel approach to study number conservation of CAs presented in [H2]. Indeed, the split-and-perturb decomposition of number-conserving local rules provides a practical way to enumerate all

two-dimensional quaternary NCCAs with the von Neumann neighborhood without having to search the entire space of $4^{4^5} \approx 3.2 \cdot 10^{616}$ two-dimensional CAs: it is sufficient to generate all split functions (2625 ones) and then for each split function to find all possible perturbations, which resulted in finding 17 582 011 number-conserving local rules. Finally, we rely on the fact that it is easy to verify that a given CA is not reversible by identifying two configurations that have the same successor. In this way, we reject from our list of two-dimensional four-state NCCAs with the von Neumann neighborhood those that are definitely not reversible. Because we have exactly 65 ones left (as constructed in the first step), we may conclude that we have found all of them.

Although the method introduced in the paper [H6] use the decomposition theorem only to overcome technical limitations of computers, I believe that it can be use to check weather the following conjecture is true or not:

Conjecture 1 *Let $d \geqslant 1$. There are exactly $(2d+1)(6d+1)$ reversible quaternary d-dimensional NCCAs with the von Neumann neighborhood.*

[H7]

The paper *"Two-dimensional rotation-symmetric number-conserving cellular automata"* [70] is a significant step forward in the study two-dimensional NCCAs with the von Neumann neighborhood, whose local rule satisfies rotation symmetry property.

Definition 15 *A two-dimensional CA with the von Neumann neighborhood is called* rotation-symmetric *if its local rule f satisfies for any $u, l, c, r, d \in Q$*

$$f\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = f\begin{pmatrix} & r & \\ u & c & d \\ & l & \end{pmatrix}. \tag{8}$$

The first characterization of two-dimensional rotation-symmetric NCCAs with the von Neumann neighborhood (R-NCCAs) was obtained by Imai et al. [26]. Their results were based on the method introduced by Tanimoto and Imai [25] allowed to describe two-dimensional number-conserving CAs with the von Neumann neighborhood in terms of *flow functions* (in the vertical, horizontal or diagonal direction) and allows to create two-dimensional number-conserving CAs.

The considerations of Imai et al. [26] allowed to enumerate all two-dimensional R-NCCAs up to five state. Moreover, the authors of [26] show that none of them is Strongly Turing Universal (*i.e.*, can simulate any Turing Machine starting from a finite configuration). On the other hand for over ten years there exists a construction of a logically universal (*i.e.*, capable of simulating any circuit performing any computation) R-NCCA with fourteen states [25]. A natural urge to close this gap arises. Of course, our inability to construct an R-NCCA with less than fourteen states does not prove there does not exist any. We may start from the opposite side and try to find a universal R-NCCA with six states or argue that such CA does not exist (and then consider seven states and so on). To do this, it would be convenient to have a complete list of R-NCCAs with a given number n of states and check them for universality. Until [H7], there is no tool allowing for generating such a list for $n > 5$. There is even no clue on how many such rules exist. The main reason for this difficulty is that the set of all two-dimensional rotation-symmetric CAs with six states and the von Neumann neighborhood is huge; it has a cardinality greater than $6^{6^5/4}$ (this number has 1513 digits). To retrieve the number-conserving CAs in this set, it cannot be searched through using the necessary and sufficient conditions given, for example, in [22].

In [H7], the same approach as in [H1] is used to establish a method that allows for the efficient enumeration of multi-state two-dimensional R-NCCAs. For example, the developed tools allow to manually find all two-dimensional R-NCCAs with six states:

Theorem 16 *There are exactly 116 two-dimensional R-NCCAs with state set $Q = \{0, 1, 2, 3, 4, 5\}$.*

Furthermore, the method is not difficult to implement on a computer and thus list all two-dimensional seven- or eight-state R-NCCAs. This is a serious step forward in trying to answer the question: what is the minimum number of states needed to obtain a universal R-NCCA. Having complete lists of rules, the experts on universality may check them one by one.

At the stage of submitting for publication, one of the reviewers asked if it was possible to answer the question "And how is it for other state sets Q?" His/her question was related to an unresolved problem:

Question 1 *Is every two-dimensional R-NCCA with n states isomorphic to some R-NCCA with state set $\{0, 1, \ldots, n-1\}$?*

In the case of two-dimensional CAs with the von Neumann neighborhood, the term "isomorphic" is understood in the following sense.

Definition 17 *We say that a CA with state set Q and local rule f is isomorphic with a CA with state set $\widetilde{Q}$ and local rule $\widetilde{f}$ if there exists a bijection $\phi : Q \to \widetilde{Q}$ such that for any $u, l, c, r, d, x \in Q$*

$$f\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = x \quad \text{if and only if} \quad \widetilde{f}\begin{pmatrix} & \phi(u) & \\ \phi(l) & \phi(c) & \phi(r) \\ & \phi(d) & \end{pmatrix} = \phi(x). \tag{9}$$

It was known that in the case $n \leqslant 4$ the answer to Question 1 is positive. Moreover, it was also known that unlike the case of two-dimensional R-NCCAs with at most four states, the number of R-NCCAs with a five-element state set depends on the structure of it. Imai et al. [26] proved that if the elements of the considered state set Q form an arithmetic progression, then there are exactly four R-NCCAs with this state set (and they are isomorphic with R-NCCA with state set $\{0, 1, 2, 3, 4\}$); otherwise, there is only one R-NCCA: the identity rule.

It took some time to answer the reviewer's question (hence the final publication date is so late), but I was able to solve this problem in detail and I got the characterization in terms of the following notation.

Definition 18 *A finite set $Q \subset \mathbb{R}$ is called m-arithmetic if some of its m elements can be ordered in an arithmetic progression.*

The obtained characterization is summarized in the theorems below.

Theorem 19 *Let a six-element state set $Q \subset \mathbb{R}$ be 6-arithmetic. There are exactly 116 R-NCCAs with the state set Q. Each of them is isomorphic with an R-NCCA with the state set $\{0, 1, 2, 3, 4, 5\}$.*

Theorem 20 *Let $Q \subset \mathbb{R}$ have six elements. If Q is not 5-arithmetic, then there exists only one R-NCCA with the state set Q, namely the identity rule.*

Theorem 21 *Let $Q \subset \mathbb{R}$ have six elements. If Q is 5-arithmetic but not 6-arithmetic, then there exist exactly two R-NCCAs with the state set Q and both of them are isomorphic with an R-NCCA with the state set $\{0, 1, 2, 3, 4, 5\}$.*

I am very glad that now the answer to Question 1 in the case $n = 6$ in known (and is positive), in particular that the last investigation made by my research team have shown that already for $n = 7$ the answer is negative (see [71]).

[H8]

The paper *"Three-dimensional rotation-symmetric number-conserving cellular automata"* [72] presents results of studying three-dimensional rotation-symmetric cellular automata with the von Neumann neighborhood that conserve the sum of states. It is shown that any such non-trivial automaton requires at least seven states, which agrees with intuition based on the known results for the one- and two-dimensional cases. Also a full characterization of these cellular automata with a seven-element state set is given and the result is quite surprising.

The vast majority of results on number conservation of CAs concerns the one-dimensional and two-dimensional cases. This is obviously connected with the double exponential increase of the number of possible local rules: if the dimension of the space is d, then the number of cells forming the von Neumann neighborhood equals $2d + 1$, thus there are $n^{n^{2d+1}}$ n-state CAs in this case.

One of the examples of rotation-symmetric NCCAs that works in any dimension is the identity, but it is, of course, the most incurious one. The aim of the paper [H8] was to answer the following two questions: what is the minimal number of states that admits the existence of a non-trivial three-dimensional rotation-symmetric NCCA (Q1) and how large is the family of all such automata with that minimal state set (Q2).

In one dimension, the von Neumann neighborhood is simply the classical neighborhood with radius one and being rotation-symmetric means being amphichiral, *i.e.*, invariant under reflection (see, e.g. [37]). It is known that among the Elementary CAs there are only five number-conserving ones: the identity rule (ECA 204), the shift-left rule (ECA 170), the shift-right rule (ECA 240), the traffic-right rule (ECA 184) and the traffic-left rule (ECA 226), but only the identity rule is also amphichiral [37]. Thus, two states is not sufficient to allow for some non-trivial rotation-symmetric NCCA in the case of one dimension. On the other hand, Boccara and Fukś [21] found all 144 one-dimensional ternary NCCAs (*i.e.*, with the state set $\{0, 1, 2\}$) and among them there are exactly two that are also amphichiral, namely the identity rule and a non-trivial CA that acts as follows: in every consecutive time step each state 2 gives "one" to each state 0 in its neighborhood, while the state 1 is permanent. Thus, in one dimension, the answer to question (Q1) is "three", and the family being the answer to question (Q2) is very poor, as it consists of only one element, which is moreover not very complicated.

In two dimensions, the answers to both questions above were given by Imai et al. [26]. They proved that all two-dimensional rotation-symmetric NCCAs with the von Neumann neighborhood with four states or less are trivial and found that the minimal state set that allows to define a non-trivial such automaton must have five elements forming an arithmetic progression. Moreover, they characterized all rotation-symmetric NCCAs

with that minimal state set and obtained that there are exactly four: the identity rule and three non-trivial ones. One of them is, as expected, analogous to the one in one dimension, which means that it acts as follows: in every consecutive time step each state 4 gives "one" to each state 0 in its neighborhood, while the states 1, 2 and 3 are permanent. However, in two dimensions a new kind of non-trivial rotation-symmetric NCCA with the minimal state set appeared. Its action depends also on the states of cells lying diagonally in the neighbourhood of a central cell (see [26] for details).

Although the computational complexity of finding all three-dimensional rotation-symmetric NCCAs with a given state set until [H8] did not allow to answer questions (Q1) and (Q2), the expectations were that the answer to (Q1) is "seven" (the number of cells in the neighborhood) and the answer to (Q2) is "the same kinds of CAs as for two dimensions and maybe some extra".

In [H8], by adopting a new approach to study NCCAs, which was proposed in [H2] (the decomposition theorem) it was possible to answer to both questions. Roughly speaking, the obtained results confirm the expectation for (Q1), but the answer to (Q2) is quite surprising, as it appeared that there is only one non-trivial three-dimensional rotation-symmetric NCCA with the smallest state set and acts analogously as the one in one dimension. And more specifically, wit was proven that the smallest state set that admits a non-trivial rotation-symmetric NCCA must have seven elements that form an arithmetic progression. Moreover, for such a state set, there is only one non-trivial rotation-symmetric NCCA and it is isomorphic with the particular NCCA with the state set $\{0, 1, 2, 3, 4, 5, 6\}$ that acts as follows: the state 6 gives "one" to every state 0 in its neighborhood, while the states 1, 2, 3, 4 and 5 are permanent. Although this result is absolutely surprising, having in mind the results for lower dimensions, now it can be easily explained. Indeed, in three dimensions the group of all rotations is rich enough so that we can switch the positions of any two non-central cells in the von Neumann neighborhood, which is impossible in two dimensions.

In contrast to the usual approaches in the field of CAs, any proof in [H8] does not require any heavy computational explorations, since using the decomposition theorem overcomes most of the difficulties appearing in higher dimensions. The remaining part of the argumentation is just elementary.

4.2 Some of my other achievements

Looking at the topics of my other papers, at least three main directions of research can be distinguished (the numbering given below corresponds to "List of scientific or artistic achievements which present a major contribution to the development of a specific discipline").

Affine Continuous Cellular Automata

Continuous Cellular Automata (CCAs) can be seen as a generalization of CAs, in which time and space are still discrete, but cells can take states from some infinite (often continuous) set. One of the best-known examples of such dynamical systems are coupled map lattices. A well-known generalization of binary CAs to CCAs with $[0, 1]$ as state set is obtained by "fuzzification" in [73] and was further studied, for example, in [74, 75]. This fuzzification process allows to associate with every binary CA some particular CCA (a so-called Fuzzy CA) through an extension of the domain of the local rule. The relationship between Fuzzy CAs and Elementary CAs (ECAs) can be found in [76].

A further generalization of the above idea has resulted in the definition of affine CCAs (ACCAs) [77]. This kind of CCAs is considered to be the simplest possible generalization of binary CAs – of course, apart from the Fuzzy CAs – as they have a local rule that is affine in each variable. Recall that a function $f(x_1, \ldots, x_n)$ is affine in the variable x_i when, for any fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, the function $f_i\colon [0, 1] \to [0, 1]$ given by $f_i(x_i) = f(x_1, \ldots, x_n)$ is affine, *i.e.* $f_i(x_i) = ax_i + b$ for some $a, b \in \mathbb{R}$.

My main results in the fild of ACCAs are contained in the following four papers (in chronological order):

[N12] Barbara Wolnik, Marcin Dembowski, Witold Bołt, Jan M. Baetens, Bernard De Baets: *Density-conserving affine continuous cellular automata solving the relaxed density classification problem*, Journal of Physics A-Mathematical and Theoretical, vol. 50, nr 34, 2017, 345103 (18 pages), 30 points, IF(1.963)

I am the author of the proofs of all theorems, in regards to the analytical layer. The remaining results and the preparation of the manuscript is the results of joint work.

[N11] Marcin Dembowski, Barbara Wolnik, Witold Bołt, Jan M. Baetens, Bernard De Baets: *Affine continuous cellular automata solving the fixed-length density classification problem*, Natural Computing, vol. 17, nr 3, 2018, pp. 467-477, 25 points, IF(1.33)

I am the author of the proofs of all theorems, in regards to the analytical layer. The remaining results and the preparation of the manuscript is the results of joint work.

[N9] Marcin Dembowski, Barbara Wolnik, Witold Bołt, Jan M. Baetens, Bernard De Baets: *Two-dimensional Affine Continuous Cellular Automata solving the relaxed density classification problem, Journal of Cellular Automata*, vol. 14, nr 3-4, 2019, pp. 191-212, 40 points, IF(0.596)

I am the author of the proofs of all theorems, in regards to the analytical layer. The remaining results and the preparation of the manuscript is the results of joint work.

[N4] Barbara Wolnik, Marcin Dembowski, Antoni Augustynowicz, Bernard De Baets: *A complete description of the dynamics of legal outer-totalistic affine continuous cellular automata*, Nonlinear Dynamics, 2022, nr online first, s.1-22. DOI:10.1007/s11071-022-07642-w 140 points, IF(5.741)

The topic of the paper was proposed by me. I am the author or the co-author of the proofs of all theorems, in regards to the analytical layer (I estimate my contribution to this part to be at least 70%*). The remaining results and the preparation of the manuscript is the results of joint work.*

Papers [N12], [N11] and [N9] concern the well-known Density Classification Problem (DCP). After DCP was introduced by Gács *et al.* [78], it was investigated more elaborately by Packard [79]. Essentially, this problem boils down to a quest for a binary CA that is able to determine whether the total number of 1s in its initial configuration is greater than the total number of 0s, in such a way that it evolves to a homogeneous configuration of 1s, and 0s otherwise. In [38, 80], however, it is shown that there is no CA that solves this problem correctly for any number of cells.

Yet, certain CAs are capable of solving this problem for many initial configurations. One of the best-known ones – the Gacs-Kurdyumov-Levin (GKL) rule [78] – leads to a correct solution for more than 80% of the initial configurations, while similar CAs were found using genetic algorithms [81, 82]. In order to obtain an infallible classifier, many scholars modified the original problem by allowing two CAs (instead of one) [83], embedding memory [84], or using probabilistic CAs [85–87]. In other studies, the output specification of the problem is changed [88], by accepting heterogeneous configurations that nonetheless obey certain properties as valid answers. Yet, this implies that multiple time steps (and memory) are needed to read the answer, or that the states of multiple cells are accessed (global access).

The search for an infallible classifier applicable to all initial configurations has also given rise to an approach that involves Continuous CAs (CCAs). For example, Briceño *et al.* [89] propose a large diffusion and small amplification CA achieving a correct solution of the DCP for any number of cells. Yet, it comes along with a parameter based on the number of cells, thus one ends up with a family of CCAs rather than a single CCA.

I started with one-dimensional ACCAs with radius one and the first question I wanted to answer was: can any such ACCA be a perfect solution to the DCP? The meaning of DCP for one-dimensional CCAs with radius one is as follows.

Definition 22 (Density classification problem in the classical sense) *Let $\rho_0 \in (0,1)$ and A_f be the global rule of a one-dimensional CCA with radius one. We say that this CCA* solves the DCP at threshold ρ_0 in the classical sense *if for any $N \in \mathbb{N}$ and $\mathbf{x} \in \{0,1\}^N$, it holds that:*

$$\begin{aligned} \rho(\mathbf{x}) < \rho_0 \quad &\Rightarrow \quad \lim_{t\to\infty} A_f^t(\mathbf{x}) = (0,0,\ldots,0)\,, \\ \rho(\mathbf{x}) > \rho_0 \quad &\Rightarrow \quad \lim_{t\to\infty} A_f^t(\mathbf{x}) = (1,1,\ldots,1)\ . \end{aligned}$$

Note that threshold $\rho_0 = \frac{1}{2}$ distinguishes if there are more 1s or 0s in the configuration. In the case of CCAs we use the following additional condition:

Definition 23 (Threshold-conserving CCA) *Let $\rho_0 \in (0,1)$ and A_f be the global rule of a CCA. We say that this CCA* conserves the threshold ρ_0 *if for any $N \in \mathbb{N}$ and $\mathbf{x} \in \{0,1\}^N$, it holds that:*

$$\begin{aligned} \rho(\mathbf{x}) < \rho_0 \quad &\Rightarrow \quad \rho\left(A_f(\mathbf{x})\right) \leqslant \rho_0\,, \\ \rho(\mathbf{x}) > \rho_0 \quad &\Rightarrow \quad \rho\left(A_f(\mathbf{x})\right) \geqslant \rho_0\ . \end{aligned}$$

One of my results in [N9] states that the answer to the above question is negative:

Theorem 24 *Let $\rho_0 \in (0,1)$. There exists no one-dimensional ACCA with radius one that solves the DCP at threshold ρ_0 in the classical sense and that conserves the threshold ρ_0.*

It turned out, however, that by introducing a slight modification of the output specification we obtained a new version of DCP, which can be solved by many ACCAs (for any initial configuration, regardless of the number of cells).

Definition 25 (Relaxed formulation of the density classification problem) *Let $\rho_0 \in (0,1)$ and A_f be the global rule of an ACCA. We say that this ACCA solves the DCP at the threshold ρ_0 in the* relaxed sense *if for all $N \in \mathbb{N}$ there is T such that for all $\mathbf{x} \in \{0,1\}^N$ and $t \geqslant T$ it holds that:*

$$\rho(\mathbf{x}) < \rho_0 \Rightarrow A_f^t(\mathbf{x}) \in [0, \rho_0)^N$$
$$\rho(\mathbf{x}) > \rho_0 \Rightarrow A_f^t(\mathbf{x}) \in (\rho_0, 1]^N$$

So, if the density of an initial configuration $\mathbf{x}$ is smaller (resp. greater) than ρ_0, then all values x_i^t are smaller (resp. greater) than ρ_0, for t large enough.

In [N12] the detailed characterization of one-dimensional density-conserving ACCAs with radius one is given. Moreover, this paper contains also an exhaustive description of the dynamics of such CAs, which shows that almost all of them solve the relaxed DCP.

In [N11] I concentrate on the classical DCP and I look for ACCAs that can solve the problem for a fixed number of cells N. (The latter assumption is motivated by the fact that the number of local elements in any real-world problem is finite, and often even known.) The main results of this paper touch two aspects. First of all, necessary conditions are derived, significantly limiting the search space of potential solutions, thus allowing to build an effective search algorithm. Moreover, a stopping criterion for a search algorithm, verifying whether a given rule is a solution, is given. The rest of the article is for demonstration purposes and shows how, thanks to these tools, it was possible to find classic DCP solutions for $N = 23$.

The next paper ([N9]) contains a discussion about two-dimensional Affine Continuous Cellular Automata with the von Neumann neighborhood in the context of solving the relaxed DCP. We focus on finding the most performant rules solving this problem among the density-conserving ones by evaluating ACCAs experimentally for a predefined set of initial configurations.

In [N4], a complete description of the dynamics of legal outer-totalistic ACCAs is given. It appeared that on the one hand, such CAs are the simplest generalization of Elementary Cellular Automata, while on the other hand, they are dynamical systems that exhibit some properties that do not occur in the binary case. Thanks to massive numerical simulations, we have been able to partition the rule space in a number of classes with a distinct behavior. A unique combination of computer simulations (sometimes quite advanced) and a panoply of analytical methods allow to lay bare the dynamics of each and every one of these continuous cellular automata and confirm all results theoretically.

The results obtained show that in the set of ACCAs considered, one can observe all types of sensitivity:

- Sensitivity to the change of a single value in an initial configuration.
- Sensitivity to the change of the number of cells in the grid.
- Sensitivity to slight changes in the parameters of a local rule.

Cellular Automata on triangular or hexagonal grids

The triangular tiling of a plane was first considered by Bays [90] at the end of the last century. Triangular CAs regularly appear in different contexts (see, for instance, [91–94]). I focus on the simplest triangular CAs: automata that update the states of the cells on the basis of the states of the adjacent cells only. Even for this case, there were no satisfactory results for the most classical CAs; moreover, some of the results that can be found in the literature (and in reputable journals) are incorrect. The same is true for hexagonal grids.

My main results in the field of CAs on triangular or hexagonal grids are contained in the following papers (in chronological order):

[N10] Antoni Augustynowicz, Jan M. Baetens, Bernard De Baets, Adam Dzedzej, Anna Nenca, Barbara Wolnik: *A note on the reversibility of 2D cellular automata on hexagonal grids*, Journal of Cellular Automata, vol. 13, nr 5-6, 2018, pp. 521-526, 15 points, IF(0.776)

This paper was created as a result of joint discussions. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 30%.

[N7] Barbara Wolnik, Maciej Dziemiańczuk, Bernard De Baets: *Recurrent misconceptions in the study of CA reversibility on triangular grids*, International Journal of Bifurcation and Chaos, vol. 31, nr 1, 2021, 2150014 (6 pages), 70 points, IF(2.836)

This paper was created as a result of joint discussions. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 50%.

[N6] Barbara Wolnik, Antoni Augustynowicz, Maciej Dziemiańczuk, Bernard De Baets: *Reversibility of non-saturated linear cellular automata on finite triangular grids*, Chaos, vol. 31, nr 1, 2021, 013136 (9 pages), 140 points, IF(3.642)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs. The preparation of the manuscript is the results of joint work.

[N1] Barbara Wolnik, Anna Nenca, Bernard De Baets: *A decomposition theorem for number-conserving multi-state cellular automata on triangular grids*, Theoretical Computer Science, Elsevier B.V., vol. 953, 2023, nr 113795, pp. 1-17, DOI:10.1016/j.tcs.2023.113795, 100 points, IF(1.002)

The topic of the paper was proposed by me. I am the author of all the theorems with the proofs. The preparation of the manuscript is the results of joint work.

The papers [N10] and [N7] contain counterexamples showing that the reversibility problem of two-dimensional linear CAs on finite hexagonal or triangular grids remains unresolved. In the later case, both for null and periodic boundary conditions. Additionally, [N7] contains a conjecture suggesting the correct characterization of the reversibility in the case of null boundary conditions.

In [N6] I propose a novel method to study the reversibility of a particular type of linear CA on triangular grids with null boundary conditions (LCAs), called *non-saturated*, for which at least one coefficient in the expression of the local rule equals zero. It turns out that this method is both extremely simple and efficient. In particular, I was able to prove the conjecture put forward in [N7], which correctly characterizes the reversibility of the LCAs studied in [95]. Moreover, my method is not limited to a very narrow class of triangular grids having $\mathbb{Z}_3$ as state set (like in [95]), but works for any regular triangular grid and any finite field.

In [N1] we study number conservation of two-dimensional CAs defined on a regular triangular grid that update the states of the cells on the basis of the states of the adjacent cells only. Even for this case, there are no satisfactory results for the most classical CAs: the so-called k-ary CAs, *i.e.*, CAs with the state set $Q = \{0, 1, 2, \ldots, k-1\}$, where k is some positive integer greater than 1. In particular, till now, there are no tools to count all such CAs. In order to study these triangular NCCAs, we adapted the new approach presented in [30] ([H2]) for NCCAs defined on a regular square grid. We showed that adjusting this line of reasoning to the setting of triangular grids leads to the full solution of the problem of finding all k-ary triangular NCCAs, regardless of the value of k.

It is worth emphasizing that in general enumerating of all NCCAs in a given setting is a real challenge. For example, even in the simplest case of one-dimensional CAs with radius one little is known. Until recently, full lists were available only for binary, ternary and quaternary such NCCAs, *i.e.*, for the state sets $\{0, 1\}$, $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ (see, for example, [12]). Recently, a complete list of 1 876 088 314 quinary NCCAs has been found (this was done using a computer and it was made possible thanks to the split-and-perturb decomposition also used in the present paper) [30]. There are no results for the state sets $\{0, 1, 2, \ldots, k-1\}$ for $k > 5$.

For the square grid with the von Neumann neighborhood in two dimensions, even if we assume rotation symmetry (a very natural assumption when modelling various phenomena, especially physical ones), the situation does not look any better. The complete list for the state set $\{0, 1, 2, 3, 4\}$ appeared in 2015 (see [26]), while for the state set $\{0, 1, 2, 3, 4, 5\}$ this was only done in 2021, as a result of very technical and tedious considerations (see [70]). That approach has been combined with the split-and-perturb decomposition, which allowed to find the complete list of rotation-symmetric NCCAs for the state set $\{0, 1, 2, 3, 4, 5, 6\}$, unfortunately, only using a computer (the list is available as a data set). So far, a similar attempt for the state set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is still beyond the computational capacity of computers. For the square grid with the Moore neighborhood, there is even no list of all NCCAs in the binary case.

From the above review, it should be clear that the enumeration of all k-ary triangular NCCAs, regardless of the value of k, presented in [N1], is something quite unexpected. In essence, this enumeration was only made possible thanks to additional properties of the split-and-perturb decomposition adjusted to the two-dimensional triangular grid, which unfortunately do not hold when considering other grids or other dimensions, making this paper indeed quite a unicum.

Stochastic Cellular Automata

I dealt with Stochastic Cellular Automata in the following articles:

[M3] Witold Bołt, Barbara Wolnik, Jan M. Baetens, Bernard De Baets: *On the identification of α-asynchronous cellular automata in the case of partial observations with spatially separated gaps*, Challenging problems and solutions in intelligent systems / de Trė Guy [i in.](red.), Studies in Computational Intelligence, nr 634, 2016, Springer, ISBN 978-3-319-30164-8, pp. 23-36.

This paper was created as a result of joint discussions. I am the author of the theoretical background. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 40%.

[M2] Witold Bołt, Aleksander Bołt, Barbara Wolnik, Jan M. Baetens, Bernard De Baets: *A statistical approach to the identification of Diploid Cellular Automata*, Theory and Practice of Natural Computing: 6th International Conference, TPNC 2017 Prague, Czech Republic, December 18-20, 2017, Proceedings / Martín-Vide Carlos, Neruda Roman, Vega-Rodríguez Miguel A. (red.), Lecture Notes In Computer Science, nr 10687, 2017, Springer, ISBN 978-3-319-71068-6, pp. 37-48.

This paper was created as a result of joint discussions. I am the author of the theoretical background. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 30%.

[N8] Witold Bołt, Aleksander Bołt, Barbara Wolnik, Jan M. Baetens, Bernard De Baets: *A statistical approach to the identification of diploid cellular automata based on incomplete observations*, BioSystems, nr 186, 2019, 103976 (12 pages), 70 points, IF(1.808)

This paper was created as a result of joint discussions. I am the author of the theoretical background. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 30%.

[M1] Jakub Neumann, Mirosław Szaban, Barbara Wolnik, Witold Bołt: *Statistical approach to the binary classification problem with the use of probabilistic cellular automata*, Przegląd badań na Wydziale Matematyki, Fizyki i Informatyki Uniwersytetu Gdańskiego 2021 / Wiesław Laskowski, Marcin Marciniak, Krzysztof Szczygielski (red.), 2021, Wydawnictwo Uniwersytetu Gdańskiego, ISBN 978-83-8206-356-1, pp. 149-160

This paper was created as a result of joint discussions. I am the author of the theoretical background. The preparation of the manuscript is the outcome of joint work. I estimate my contribution to be 25%.

The research in these articles is the result of collaboration with Research Unit Knowledge-Based Systems (KERMIT) from Ghent University. My role in all these articles was to select the appropriate mathematical tools and take care of the theoretical layer.

In [M3] we present a statistical method, based on frequencies, for identifying so-called α-asynchronous Cellular Automata from partial observations, *i.e.* pre-recorded configurations of the system with some cells having an unknown (missing) state. The presented method, in addition to finding the unknown Cellular Automaton, is able to unveil the missing state values with high accuracy.

In [M2] we focus on the identification of a special class of Stochastic CAs (SCAs), called diploid CAs. The identification method presented in this paper is an extension of the method presented in [M3]. The presented results form the first step towards establishing a general identification method, based on incomplete observations, for SCAs, as any SCA can be expressed in a form of mixture of a finite number of deterministic CAs, whereas in diploid CAs we allow only for mixing two such CAs.

[N8] is a continuation of the work on the identification of diploid CAs – the case of incomplete observations was discussed. The goal of the identification algorithm is to estimate the parameters of the underlying SCA and to estimate the missing states in the observations.

In [M1] we presented a general idea of improving the accuracy of a CA-based classifier, by using statistics: by repeating the computation multiple times and considering aggregated results, we were able to improve the accuracy of the classification.

5. Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

Since 2016, I have been conducting significant scientific activity at the Ghent University (for research unit KERMIT, Department of Data Analysis and Mathematical Modelling, Faculty of Bioscience Engineering, Ghent University, Belgium). My scientific activity carried out at that institution initially consists mainly in mathematical support of research in the field of Affine Continuous Cellular Automata and Stochastic Cellular Automata carried out at KERMIT. Then was extended to other topics in cellular automata field. Additionally, during my stays at Ghent (on average once a year - with a pandemic break), I give lectures for employees and PhD students.

The dates of my stays at Ghent University:

XII 2017 - one-week scientific visit: intensive scientific work and lectures for employees and PhD students.

I 2019 - one-week scientific visit: intensive scientific work and lectures for employees and PhD students.

VI 2019 - one-week scientific visit: intensive scientific work and lectures for employees and PhD students.

X 2022 - one-week scientific visit: intensive scientific work and lectures for employees and PhD students.

Due to the pandemic, in the years 2020 – 2021, the trips were changed into a regular remote work.

Currently, my status at the Ghent University is an *affiliated professor* (https://kermit.ugent.be/affiliated.php) and in the last few papers my affiliation with Ghent University is indicated.

6. Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

Teaching

Since 1991 I have taught a variety of courses at the Gdansk University, in particular: calculus (different levels), linear algebra, functional analysis, dynamical systems, mathematical economy or probability. For my teaching work, I received the "Teacher of the Year" award named after Krzysztof Celestyn Mrongowiusz (2012), awarded for outstanding didactic achievements. I also won a distinction in this competition twice.

In addition, I have many other activities connected with teaching - below are some of them:

I am a co-author of a new specialization during the Mathematics course: Economic Mathematics; a new course: Mathematical Modeling and Data Analysis, and a new course for doctoral studies: Interdisciplinary Doctoral Studies in Mathematical Modelling (IDSMM).

I conducted seminars introducing mathematical tools for the Medical University of Gdańsk.

Since 2006, I have been the supervisor of a student science club "Kolor". During its activity "Kolor" organized and conducted many scientific events (lectures, conferences, meetings) and events popularizing mathematics (workshops, competitions).

I was teaching during the introductory courses for students starting studies in the Faculty of Mathematics, Physics and Informatics.

In the years 2017 - 2019 I was co-promoter of the Ph.D student Adam Dzedzej, who was working under supervision of Profesor Bernard De Baets (Ghent University in Belgium). The doctorate was defended with distinction (suggested by both foreign reviewers). Now I am working as co-promoter with another PhD student.

I am a co-author of several problem books for high school students and I am the author of one script for students.

For over 15 years I have been conducting regular mathematics classes for mathematically gifted students (at various levels of advancement) – currently on average 4 hours a week.

In the years 2019 - 2021, I worked individually with a student of Informatics on the cellular automata topic. He presented his results at an international conference Automata & WAN 2021 (12-17 July 2021, Marseille in France) and he is co-author of the paper "An exploration of reversible septenary number-conserving cellular automata" (submitted to Natural Computing).

Organizational achievements

My organizational activity is very wide - below are some of the most important examples:

Since 2016, I have been Deputy Dean for Students and Education of the Faculty of Mathematics, Physics and Informatics. During the academic year 2009/2010 I was Deputy Director for Education at the Institute of Mathematics.

I was the main organizer and coordinator on behalf of the Gdańsk University of the event covering the entire Pomeranian Voivodeship: 2015 the Year of Mathematics in Pomerania (The work on the preparation and realization of the initiative lasted from 2012 to 2016). (https://www.gdansk.pl/urzad-miejski/ wiadomosci/matematyka-dla-kazdego-rok-matematyki-na-pomorzu,a,38308)

Since 2016 I have been the main coordinator of the project „Zdolni z Pomorza – Uniwersytet Gdański". As a part of this activity, I am responsible for organizing various forms of support for gifted pupils from the Pomeranian Voivodeship (so far, 4 000 pupils have been supported).

I also organized or co-organized many smaller events, such as: the student's competition Mat2Tab (for the mathematical, educational software) – 3 editions; "University for graduates" – several editions; "Girls for Natural sciences" (2011); IT competition for students "3CityCup" – several editions; students IT conferences ITAD – several editions, to list just a few.

Achievements in popularization of science

I am very willing to contribute to the popularization of science and my activity in this field is very rich. Below I present the most important forms of this activity:

A lot of popular lectures for pupils (currently on average about 40 each year) on different areas of mathematics and its applications.

Several popular lectures for teachers of mathematics (mostly during some conferences for teachers).

Since 2013 I am a member of "bez rutyny" association: http://www.bezrutyny.pl/ (only in polish)– the mail goal of the association is the popularization of mathematics and other sciences among young people. As part of this activity, I am one of the leaders responsible for the coordination and organization of Pomeranian Mathematical Games – currently the seventh edition is going on. Each year there are about hundred schools taking part in the Games. Moreover, I am a co-organizer and lecturer during 15 scientific camps (Mathematical Polygons).

I am a long-time member of the District Committee of the Mathematical Olympiad.

My activities in the field of teaching, organization and popularization has been appreciated with the following awards:

2010 – The University of Gdańsk Rector's Award for preparing a grant for "Kierunki zamawiane" (second degree team award)

2014 – The University of Gdańsk Rector's Award for IDSMM project (second degree team award)

2015 – The University of Gdańsk Rector's Award for organizational activity (second degree individual award)

2018 – The University of Gdańsk Rector's Award for didactic, educational and organizational achievements (second degree individual award)

In the field of scientific work, I received the following awards:

2019 – Scientific Award of the Rector of UG named in honor of Professor Karol Taylor

2021 – The University of Gdańsk Rector's Award for scientific achievements documented by publications (second degree individual award)

7. Apart from information set out in 1-6 above, the applicant may include other information about his/her professional career, which he/she deems important.

For personal reasons, my scientific career started only in 2016. Earlier, after completing my doctorate, I had to put a stronger emphasis on caring for my children, which was incompatible with intensive research work and research trips abroad. Currently, I am a leader of a five-person team working on cellular automata (at the Gdańsk University) and also I am an affiliated professor at the Ghent University.

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